Problem 1: Let R be the region bounded by the lines $y - x = 0, y - x = 2, y + x = 0, y + x = 2$. Use a change of variables to evaluate

$$\iint_R \sqrt{y^2 - x^2}\, dA. \tag{1}$$

Solution: We use the substitution $u = y - x$ and $v = y + x$ as suggested by the defining equations of the boundary curves. We also see in the picture below that this substitution results in a simple tesselation of our region R, which shows us that the new region of integration in the uv-plane is just $R' = \{(u, v) \mid 0 \le u \le 2, 0 \le v \le 2\}$.

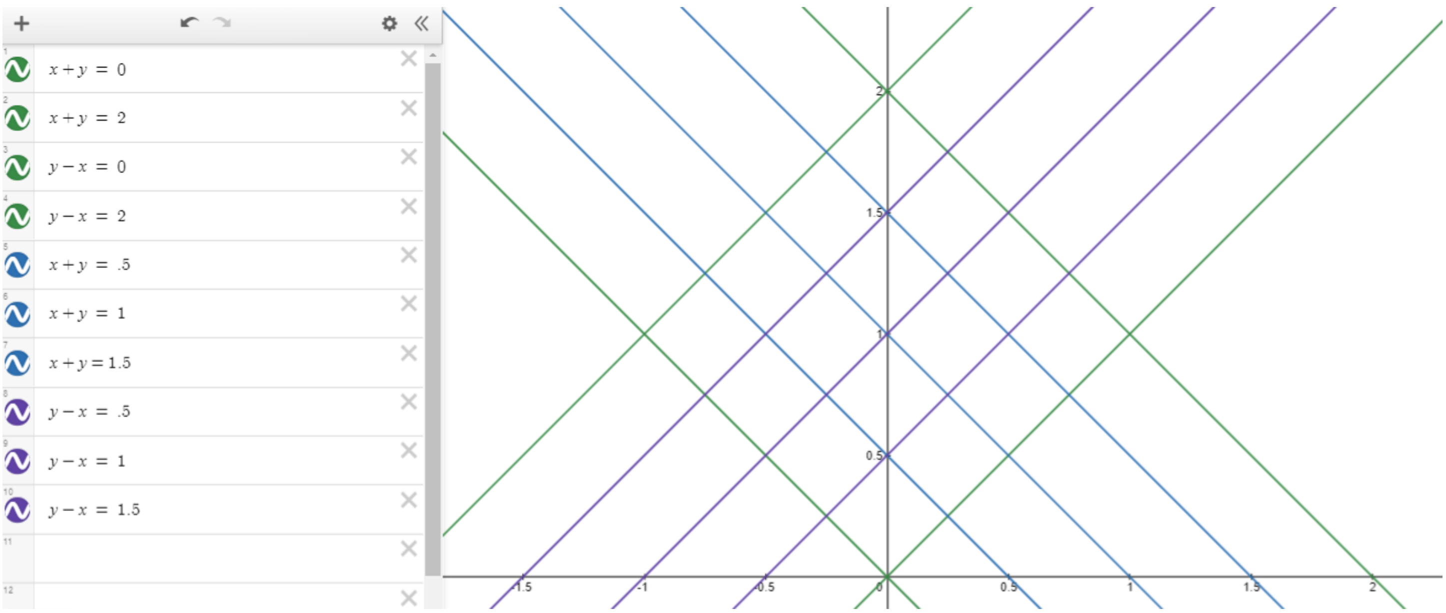


Figure 1: A picture of the region R and the tesselation that results from our given change of variables.

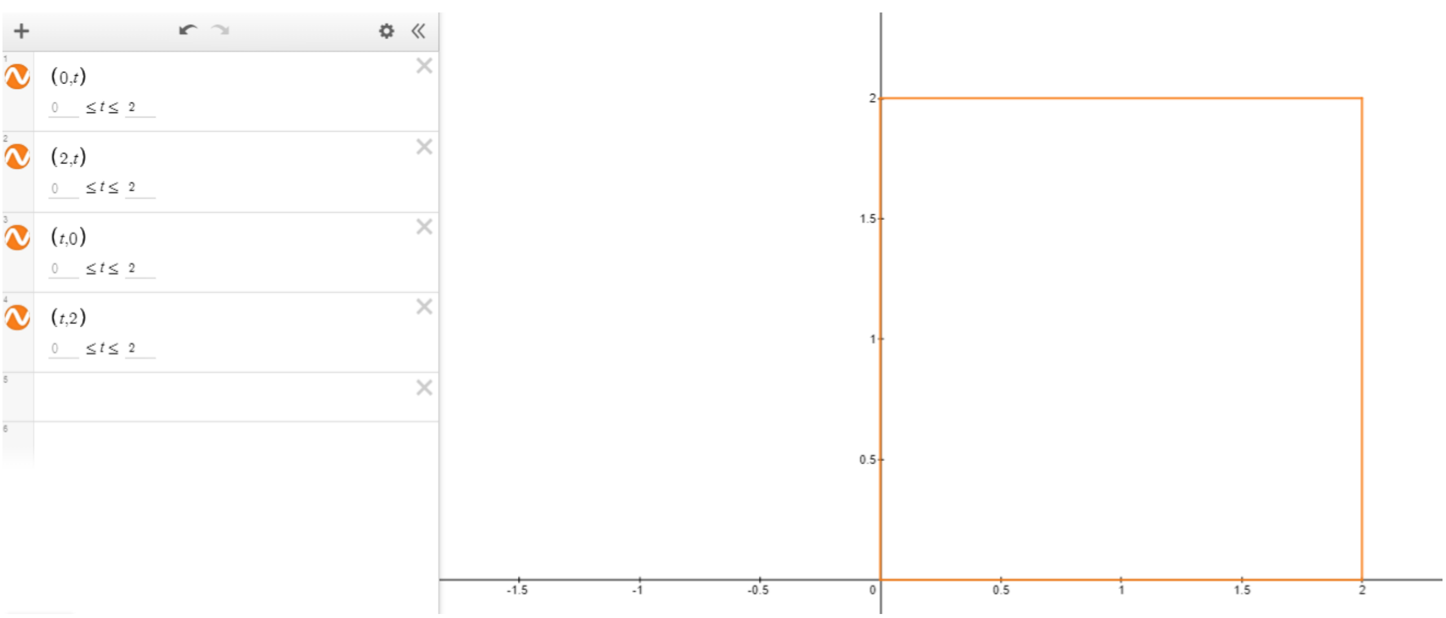


Figure 2: A picture of the region of integration in the uv-plane R'.

In order to calculate the Jacobian $J(u,v)$, we need to solve for x and y in terms of u and v. To this end, we see that

$$\begin{array}{rcl} u &=& y-x \\ v &=& y+x \end{array} \to \begin{array}{rcccl} x &=& \frac{1}{2}\Big((y+x)-(y-x)\Big) &=& \frac{1}{2}(v-u) \\ y &=& \frac{1}{2}\Big((y+x)+(y-x)\Big) &=& \frac{1}{2}(v+u) \end{array}. \tag{2}$$

We now see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = (-\frac{1}{2})\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{2}. \tag{3}$$

It follows that $|J(u,v)| = |-\frac{1}{2}| = \frac{1}{2}$. We now see that

$$\text{Area}(R) = \iint_R \sqrt{y^2-x^2}dA = \iint_{R'} \sqrt{(y-x)(y+x)}\cdot|J(u,v)|dA \tag{4}$$

$$= \int_0^2\int_0^2 \sqrt{uv}\frac{1}{2}dudv = \frac{1}{2}\int_0^2\int_0^2 u^{\frac{1}{2}}v^{\frac{1}{2}}dudv = \frac{1}{2}\int_0^2 \frac{2}{3}u^{\frac{3}{2}}v^{\frac{1}{2}}\Big|_{u=0}^2 dv \tag{5}$$

$$= \frac{1}{2}\int_0^2 \frac{2}{3}2^{\frac{3}{2}}v^{\frac{1}{2}}dv = \frac{2\sqrt{2}}{3}\int_0^2 v^{\frac{1}{2}}dv = \frac{4\sqrt{2}}{9}v^{\frac{3}{2}}\Big|_0^2 = \boxed{\frac{16}{9}}. \tag{6}$$

Problem 2: Let R be the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y = x$ and $y = 3x$. Evaluate

$$\iint_R y^4 dA. \tag{7}$$

Note that you can also solve this problem in Cartesian coordinates and polar coordinates, not just a change of variables. Try solving it with all three methods and compare their difficulties!

Solution 1: Our first solution will use a change of variables. Noting that the line $y = x$ can be rewritten as $\frac{y}{x} = 1$ and the line $y = 3x$ can be rewritten as $\frac{y}{x} = 3$, we decide to use the change of variables $u = xy$ and $v = \frac{y}{x}$ in order to make our new region of integration in the uv-place a rectangle. In particular, we see that $R' = \{(u, v) : 1 \leq u \leq 4, 1 \leq v \leq 3\}$ is the new region of integration.

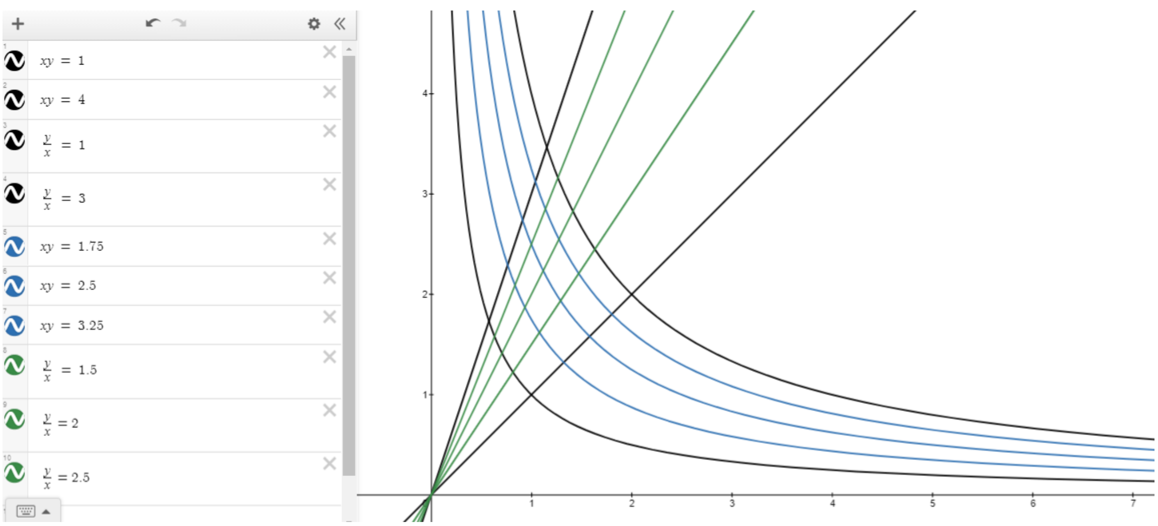


Figure 3: The original region of integration in the xy-plane R.

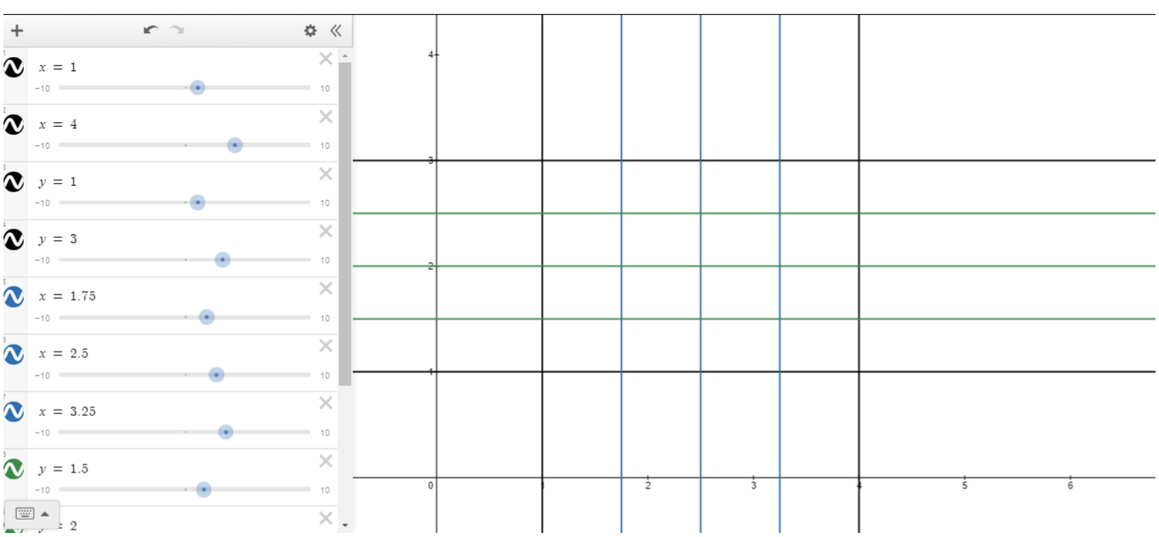


Figure 4: The new region of integration in the uv-plane R'.

In order to calculate the Jacobian $J(u,v)$ we must first solve for x and y in terms of u and v. To that end, we see that

$$\begin{aligned} u &= xy \\ v &= \tfrac{y}{x} \end{aligned} \rightarrow x = (x^2)^{\frac{1}{2}} = (\frac{u}{v})^{\frac{1}{2}} = u^{\frac{1}{2}}v^{-\frac{1}{2}} \text{ and } y = (y^2)^{\frac{1}{2}} = u^{\frac{1}{2}}v^{\frac{1}{2}}. \tag{8}$$

We note that we took the positive square roots above since we are working in the first quadrant of the xy-place, so x and y are both nonnegative. We now see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} \tag{9}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} \cdot \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} - (-\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}}) \cdot \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} = \frac{1}{2}v^{-1}. \tag{10}$$

Since $1 \le v \le 3$ in our new region of integration R', we see that $\frac{1}{2}v^{-1} \ge 0$ on R', so $|J(u,v)| = J(u,v)$ on R'. We now see that

$$\iint_R y^4 dA = \iint_{R'} (u^{\frac{1}{2}}v^{\frac{1}{2}})^4 |J(u,v)| dA = \int_1^4 \int_1^3 u^2v^2 \cdot \frac{1}{2}v^{-1} dv du \tag{11}$$

. . . .

$$= \frac{1}{2}\int_1^4 \int_1^3 u^2 v dv du = \frac{1}{2}\int_1^4 \frac{1}{2}u^2v^2\Big|_{v=1}^3 du \tag{12}$$

. . . .

$$= \int_1^4 2u^2 du = \frac{2}{3}u^3\Big|_1^4 = \boxed{42}. \tag{13}$$

Solution 2: Our next solution will use polar coordinates. We begin by observing that in the first quadrant we have

$$y = x \Leftrightarrow r\sin(\theta) = r\cos(\theta) \Leftrightarrow \sin(\theta) = \cos(\theta) \Leftrightarrow \theta = \frac{\pi}{4}, \tag{14}$$

$$y = 3x \Leftrightarrow r\sin(\theta) = 3r\cos(\theta) \Leftrightarrow \tan(\theta) = 3 \Leftrightarrow \theta = \tan^{-1}(3), \tag{15}$$

$$1 = xy = r^2\cos(\theta)\sin(\theta) \Leftrightarrow r = \sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}, \text{ and} \tag{16}$$

$$4 = xy = r^2\cos(\theta)\sin(\theta) \Leftrightarrow r = \sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}. \tag{17}$$

It follows that

$$\iint_R y^4 dA = \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \int_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}} (r\sin(\theta))^4 r dr d\theta \tag{18}$$

. . . .

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \int_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}} r^5 \sin^4(\theta) dr d\theta \tag{19}$$

. . . .

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \left(\frac{1}{6} r^6 \sin^4(\theta) \Big|_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}} \right) d\theta \tag{20}$$

. . . .

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \frac{21}{2} \frac{\sin(\theta)}{\cos^3(\theta)} d\theta = \frac{21}{2} \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \underbrace{\frac{\sin(\theta)}{\cos(\theta)}}_{u=\tan(\theta)} \cdot \underbrace{\frac{1}{\cos^2(\theta)} d\theta}_{du=\sec^2(\theta)d\theta} \tag{21}$$

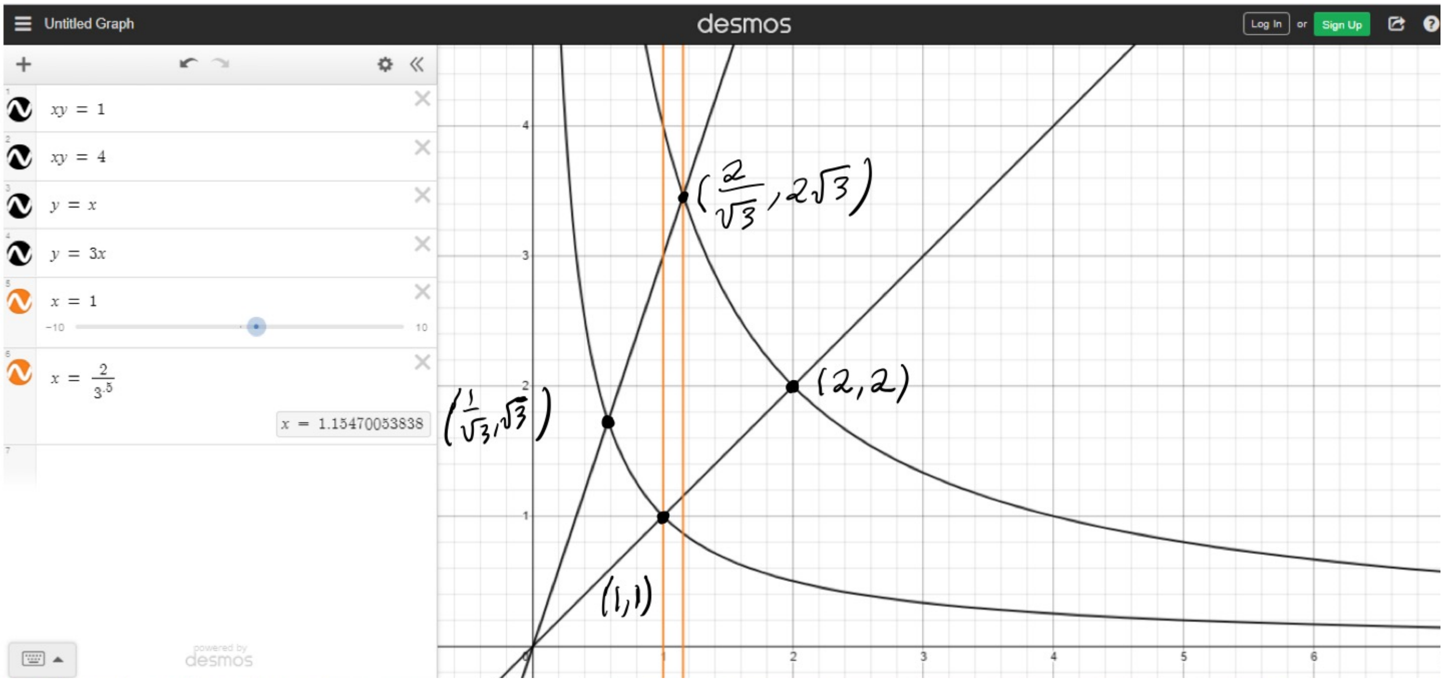
. . . .

$$\frac{21}{2} \left(\frac{1}{2} tan^2(\theta) \Big|_{\frac{\pi}{4}}^{\tan^{-1}(3)} \right) = \frac{21}{4}(3^2 - 1^2) = \boxed{42}. \tag{22}$$

Solution 3: Our last solution will use Cartesian coordinates. We begin by observing that in the first quadrant we have

$$\begin{array}{ccccccccc} xy & = & 1 & \text{and} & y & = & x & \to & (x,y) & = & (1,1) \\ xy & = & 1 & \text{and} & y & = & 3x & \to & (x,y) & = & (\frac{1}{\sqrt{3}}, \sqrt{3}) \\ xy & = & 4 & \text{and} & y & = & x & \to & (x,y) & = & (2,2) \\ xy & = & 4 & \text{and} & y & = & 3x & \to & (x,y) & = & (\frac{2}{\sqrt{3}}, 2\sqrt{3}) \end{array} \tag{23}$$

Now that we have identified the 'corners' of our region as shown in the picture below, we are able to set up and evaluate the desired double integral.



$$\iint_R y^4 dA = \int_{\frac{1}{\sqrt{3}}}^{1}\int_{\frac{1}{x}}^{3x} y^4 dydx + \int_{1}^{\frac{2}{\sqrt{3}}}\int_{x}^{3x} y^4 dydx + \int_{\frac{2}{\sqrt{3}}}^{2}\int_{x}^{\frac{4}{x}} y^4 dydx \tag{24}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{5}y^5\Big|_{y=\frac{1}{x}}^{3x} dx + \int_{1}^{\frac{2}{\sqrt{3}}} \frac{1}{5}y^5\Big|_{y=x}^{3x} dx + \int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{5}y^5\Big|_{y=x}^{\frac{4}{x}} dx \tag{25}$$

$$= \frac{1}{5}\Big(\int_{\frac{1}{\sqrt{3}}}^{1}(243x^5 - x^{-5})dx + \int_{1}^{\frac{2}{\sqrt{3}}} 242x^5 dx + \int_{\frac{2}{\sqrt{3}}}^{2}(1024x^{-5} - x^5)dx\Big) \tag{26}$$

$$= \frac{1}{5}\Big((\frac{81}{2}x^6 + \frac{1}{4}x^{-4}\Big|_{\frac{1}{\sqrt{3}}}^{1}) + (\frac{121}{3}x^6\Big|_{1}^{\frac{2}{\sqrt{3}}}) + (-256x^{-4} - \frac{1}{6}x^6\Big|_{\frac{2}{\sqrt{3}}}^{2})\Big) \tag{27}$$

$$\frac{1}{5}\Big((\frac{81}{2} + \frac{1}{4} - \frac{3}{2} - \frac{9}{4}) + (\frac{121}{3}\cdot(\frac{64}{27} - 1)) + (-16 - \frac{32}{3} + 144 + \frac{32}{81})\Big) \tag{28}$$

$$\frac{1}{5}\Big(37 + \frac{121\cdot 37}{81} + 128 - \frac{26\cdot 32}{81}\Big) = \boxed{42}. \tag{29}$$

Problem 3: Find the volume of the solid D that is bounded by the planes $y - 2x = 0, y - 2x = 1, z - 3y = 0, z - 3y = 1, z - 4x = 0$, and $z - 4x = 3$.

Solution: We use the substitution $u = y - 2x, v = z - 3y$, and $w = z - 4x$ as suggested by the defining equations of the boundary curves. We also see in the pictures below that this substitution results in a simple tesselation of our region R, which shows us that the new region of integration in the uvw-space is just $R' = \{(u, v, w) \mid 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 3\}$.

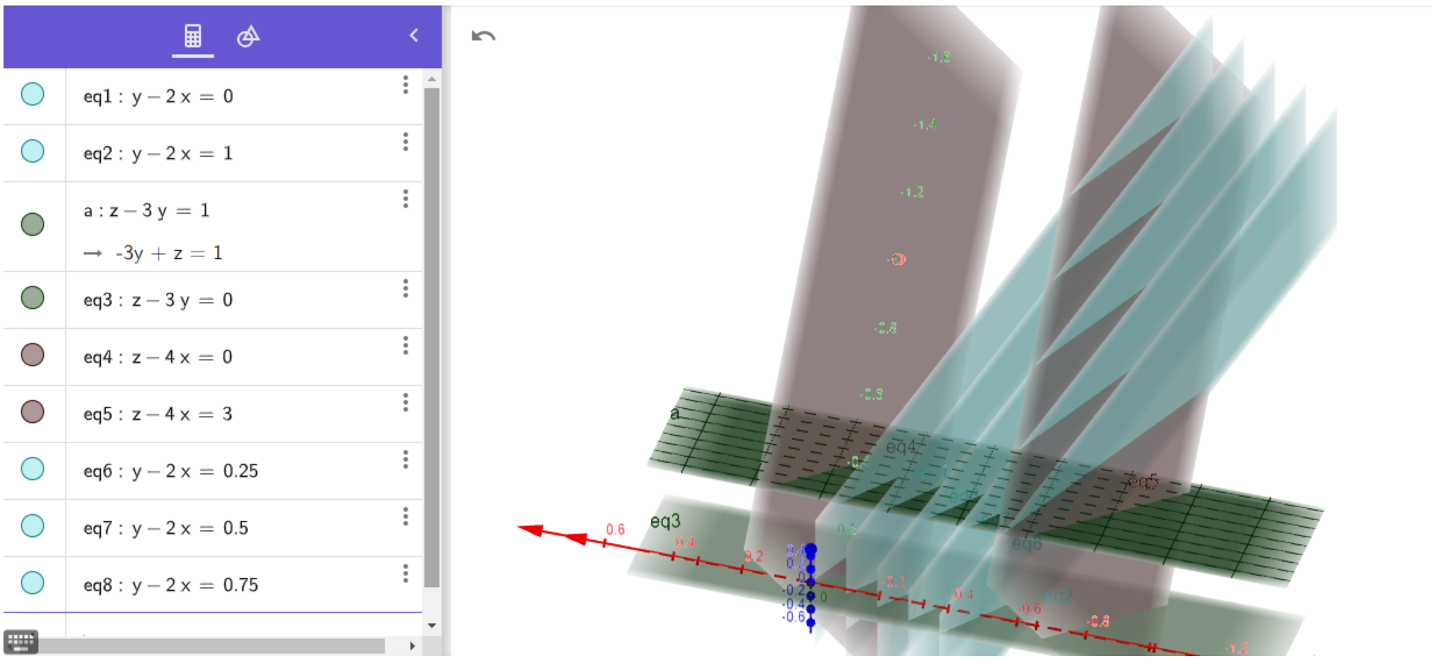


Figure 5: A visualization of the impact of changing the value of $y - 2x$.

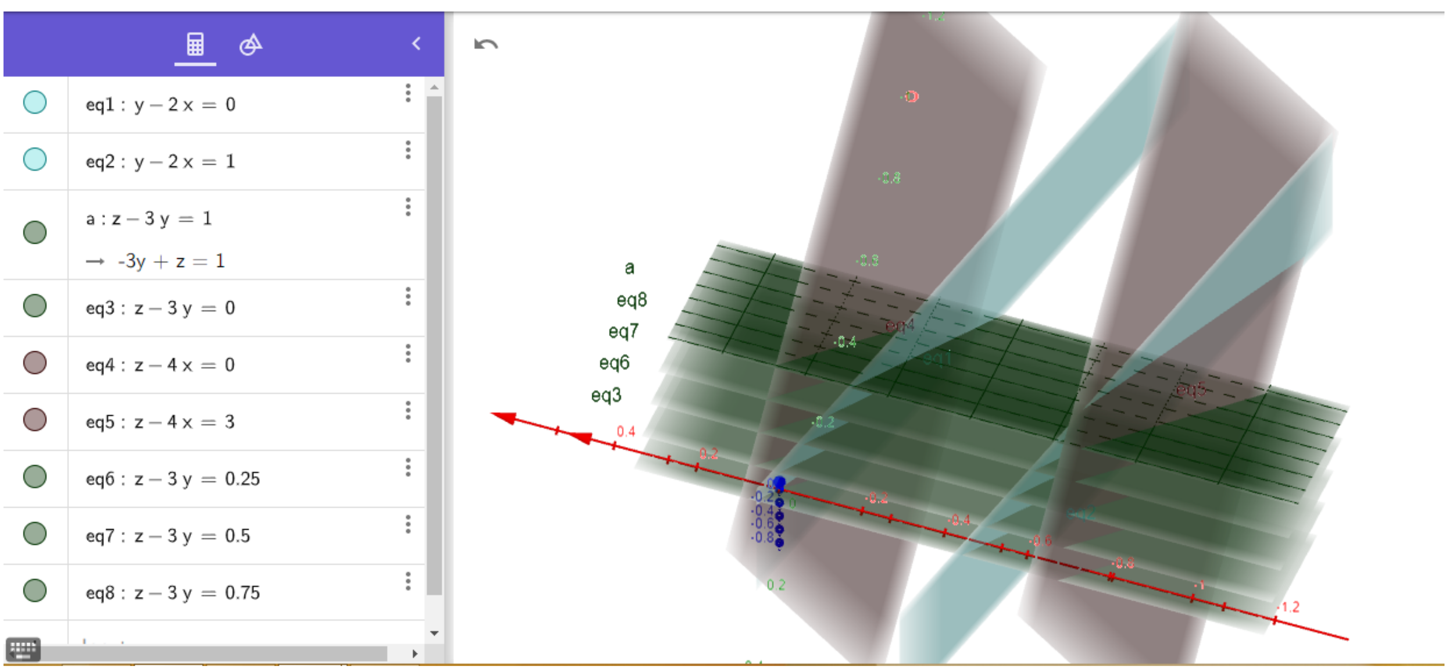


Figure 6: A visualization of the impact of changing the value of $z - 3y$.

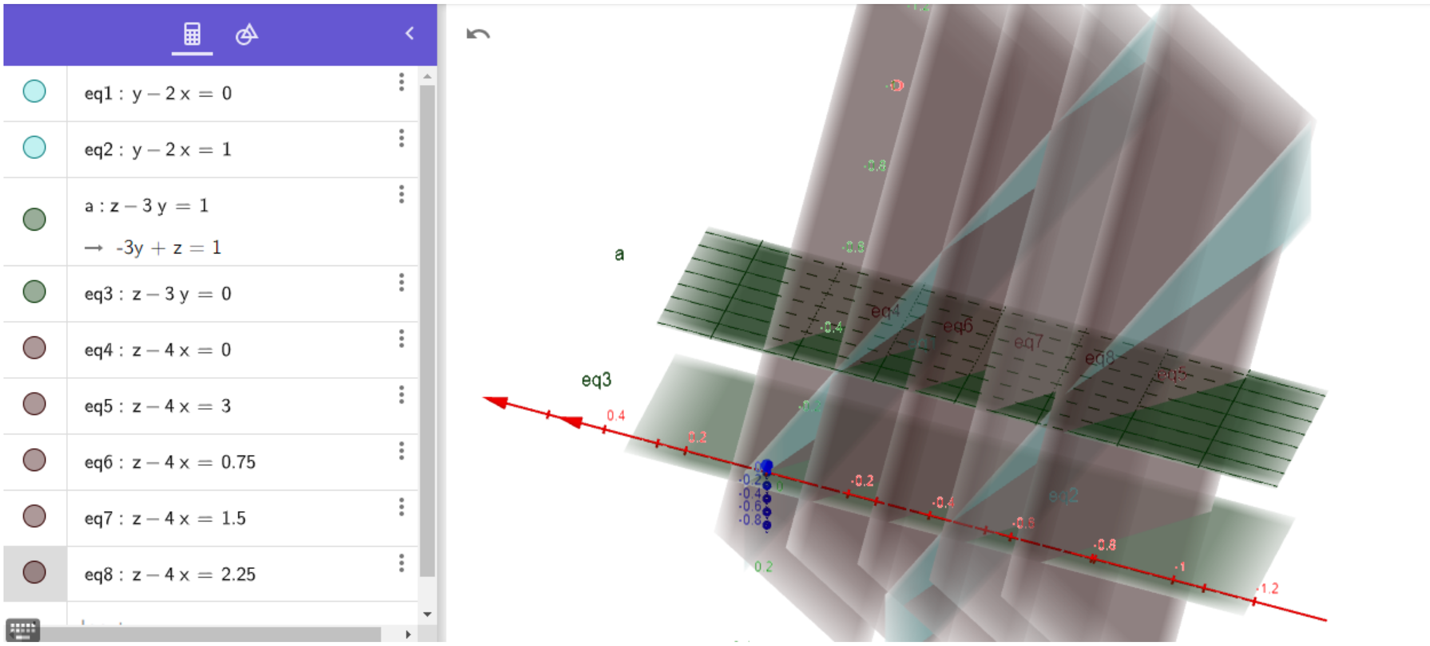


Figure 7: A visualization of the impact of changing the value of $z - 4x$.

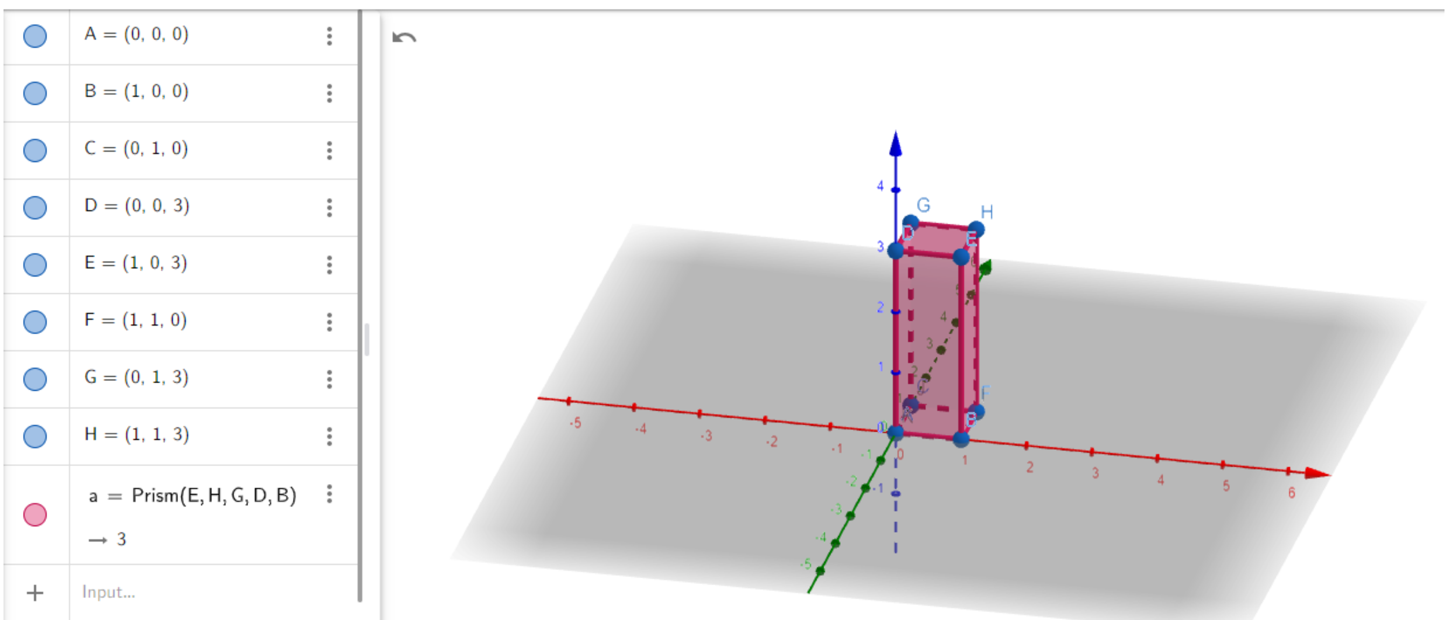


Figure 8: The region of integration R' in the uvw-space.

In order to calculate the Jacobian $J(u, v, w)$, we need to solve for x, y, and z in terms of u, v, and w. To this end, we see that

$$\begin{aligned} u &= y - 2x \\ v &= z - 3y \quad \to v - w = 4x - 3y \\ w &= z - 4x \end{aligned} \tag{30}$$

$$\to (v - w) + 3u = -2x \to x = \frac{1}{2}(-3u - v + w) \tag{31}$$

$$\rightarrow \begin{array}{ccccccc} y & = & u+2x & = & u-3u-v+w & = & -2u-v+w \\ z & = & w+4x & = & w-6u-2v+2w & = & -6u-2v+3w \end{array}. \tag{32}$$

We now see that

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix} \tag{33}$$

. . . .

$$= -\frac{3}{2}\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} - (-\frac{1}{2})\begin{vmatrix} -2 & 1 \\ -6 & 3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -2 & -1 \\ -6 & -2 \end{vmatrix} \tag{34}$$

. . . .

$$= -\frac{3}{2}(-3+2) + \frac{1}{2}(-6+6) + \frac{1}{2}(4-6) = \frac{1}{2}. \tag{35}$$

It follows that $|J(u,v,w)| = \frac{1}{2}$. We now see that

$$\text{Volume}(D) = \iiint_D 1dV = \iiint_{D'} 1 \cdot |J(u,v,w)|dV \tag{36}$$

$$= \int_0^1 \int_0^1 \int_0^3 \frac{1}{2} dudvdw = (1-0)(1-0)(3-0)\frac{1}{2} = \boxed{\frac{3}{2}}. \tag{37}$$

Problem 4: This problem has parts **a.-g.** spread out across the following pages. Your solutions to parts **a, b,** and **f** need (hand drawn or computer generated) pictures.

Consider the Transformation T from the uv-plane to the xy-plane given by $T(u,v) = (u^2 - v^2, 2uv)$.

a. Show that the lines $u = a$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices[1] on the positive x-axis.[2] Compare the images of the lines $u = a$ and $u = -a$ under T.

b. Show that the lines $v = b$ in the uv-plane map to parabolas in the xy-plane that open in the positive x-direction with vertices on the negative x-axis.[3] Compare the images of the lines $v = b$ and $v = -b$ under T.

c. Evaluate $J(u,v)$.

Solution to part a: We see that $T(a,v) = (a^2 - v^2, 2av)$. Setting $x = a^2 - v^2$ and $y = 2av$, we see that $v = \frac{1}{2a}y$, so $x = a^2 - (\frac{1}{2a}y)^2 = a^2 - \frac{1}{4a^2}y^2$, or equivalently, $x - a^2 = -\frac{1}{4a^2}y^2$. Since $a^2 > 0$ and $-\frac{1}{4a^2} < 0$ when $a \neq 0$, we see (as mentioned in the footnote) that $T(a,v)$ is the parameterization of a parabola that opens in the negative x-direction and has its vertex on the positive x-axis. We see that $T(-a,v) = (a^2 - v^2, -2av) = (a^2 - (-v)^2, 2a(-v) = T(a,-v)$, so $T(a,v)$ and $T(-a,v)$ parameterize the same parabola in the xy-plane, but the parameterizations are in opposite directions (if $a \neq 0$). We also see that $T(0,v) = (-v^2, 0)$, which is a parameterization (with repitition) of the negative x-axis, which can be viewed as a degenerate parabola that opens in the negative x-direction and has its vertex at $(0,0)$.

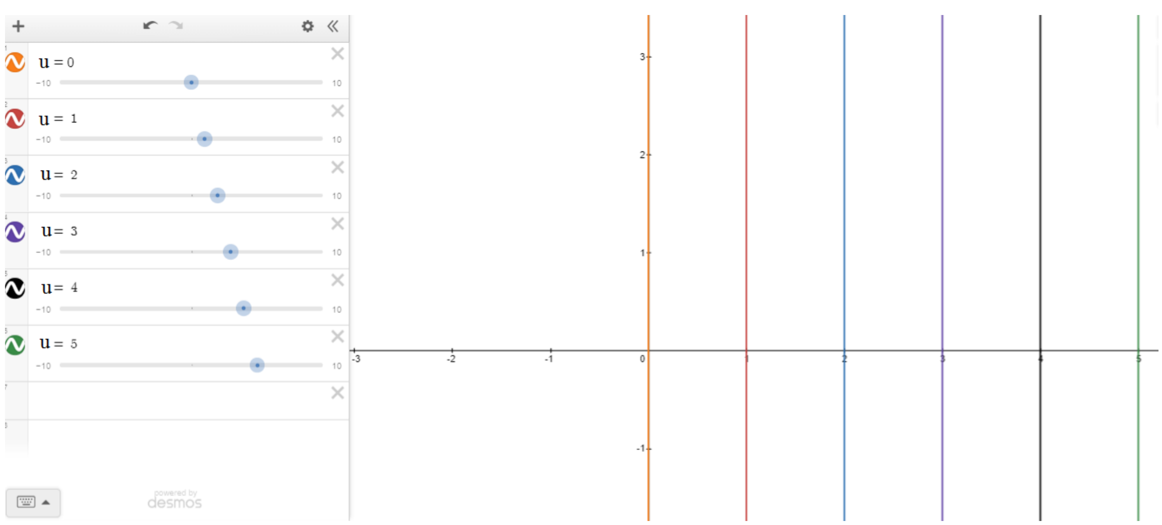


Figure 9: Vertical lines in the uv-plane.

[1] The vertex of the parabola $y = x^2$ is the point $(0,0)$ and the vertex of the parabola $x = y^2$ is also $(0,0)$.

[2] You have to show that the curve $\vec{r_1}(v) = (a^2 - v^2, 2av)$ represents the same curve as $x - c = -by^2$ for some positive numbers b and c.

[3] You have to show that the curve $\vec{r_2}(u) = (u^2 - b^2, 2ub)$ represents the same curve as $x - c = by^2$ for some positive number b and some negative number c.

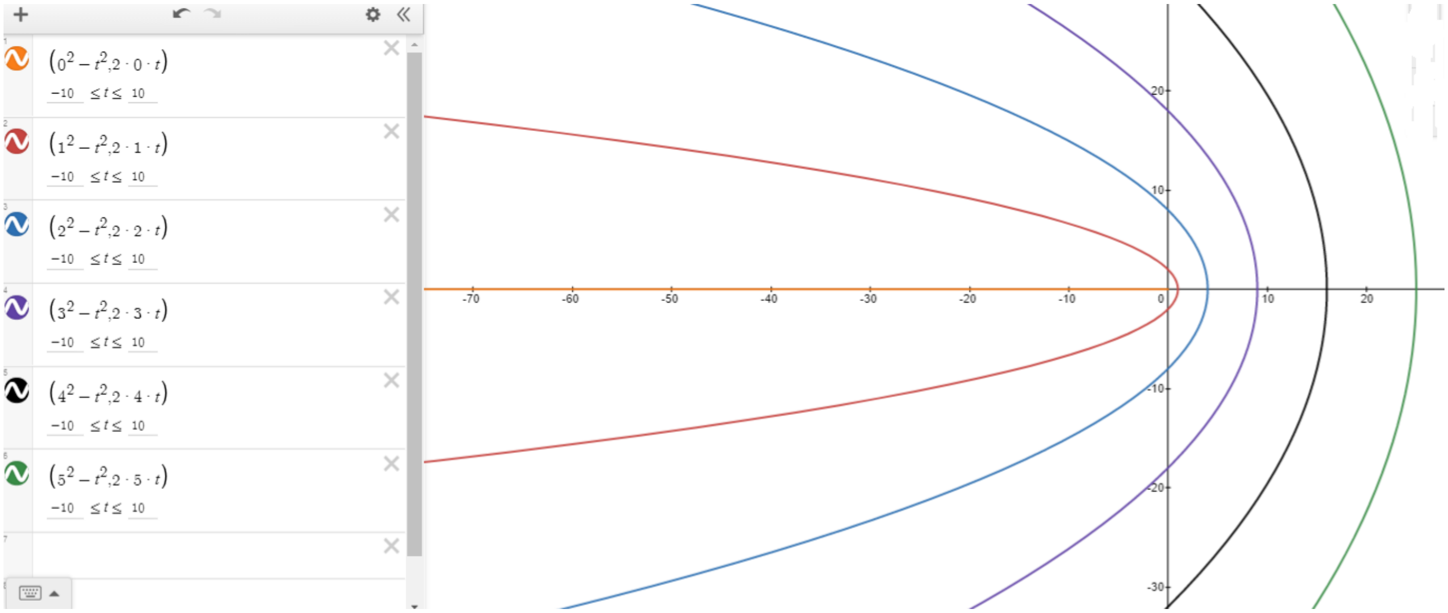


Figure 10: The parabolas in the xy-plane corresponding to vertical lines in the uv-plane under the transformation T.

Solution to part b: We see that $T(u,b) = (u^2-b^2, 2ub)$. Setting $x = u^2-b^2$ and $y = 2ub$, we see that $u = \frac{1}{2b}y$, so $x = (\frac{1}{2b}y)^2-b^2 = \frac{1}{4b^2}y^2-b^2$, or equivalently, $x-(-b^2) = \frac{1}{4b^2}y^2$. Since $-b^2 < 0$ and $\frac{1}{4b^2} > 0$ when $b \neq 0$, we see (as mentioned in the footnote) that $T(u,b)$ is the parameterization of a parabola that opens in the postive x-direction and has its vertex on the negative x-axis. We see that $T(u,-b) = (u^2-b^2, -2ub) = ((-u)^2-b^2, 2(-u)b) = T(-u,b)$, so $T(u,b)$ and $T(u,-b)$ parameterize the same parabola in the xy-plane, but the parameterizations are in opposite directions (if $b \neq 0$). We also see that $T(u,0) = (u^2,0)$, which is a parameterization (with repitition) of the positive x-axis, which can be viewed as a degenerate parabola that opens in the positive x-direction and has its vertex at $(0,0)$.

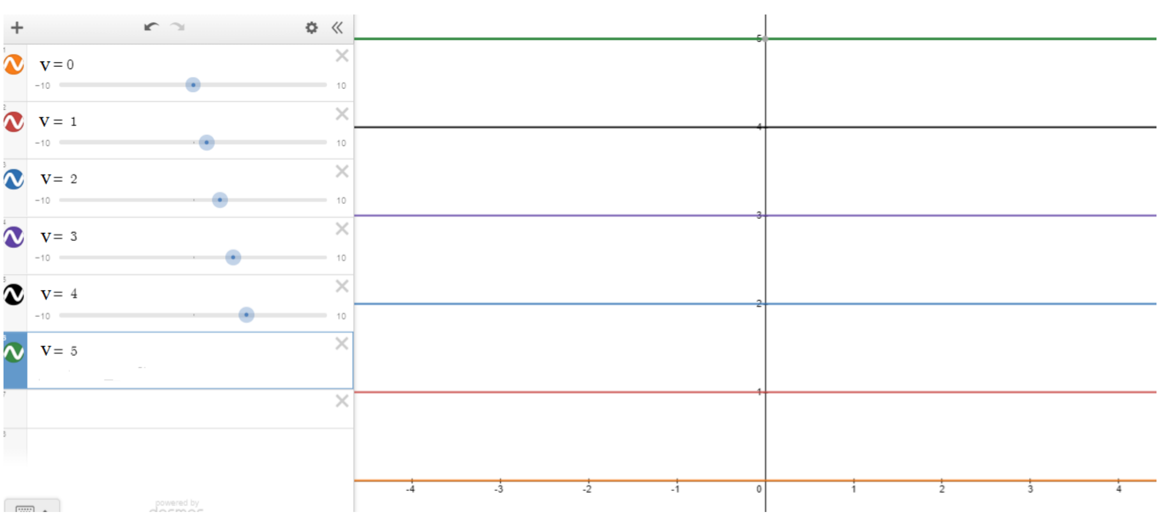


Figure 11: Horizontal lines in the uv-plane.

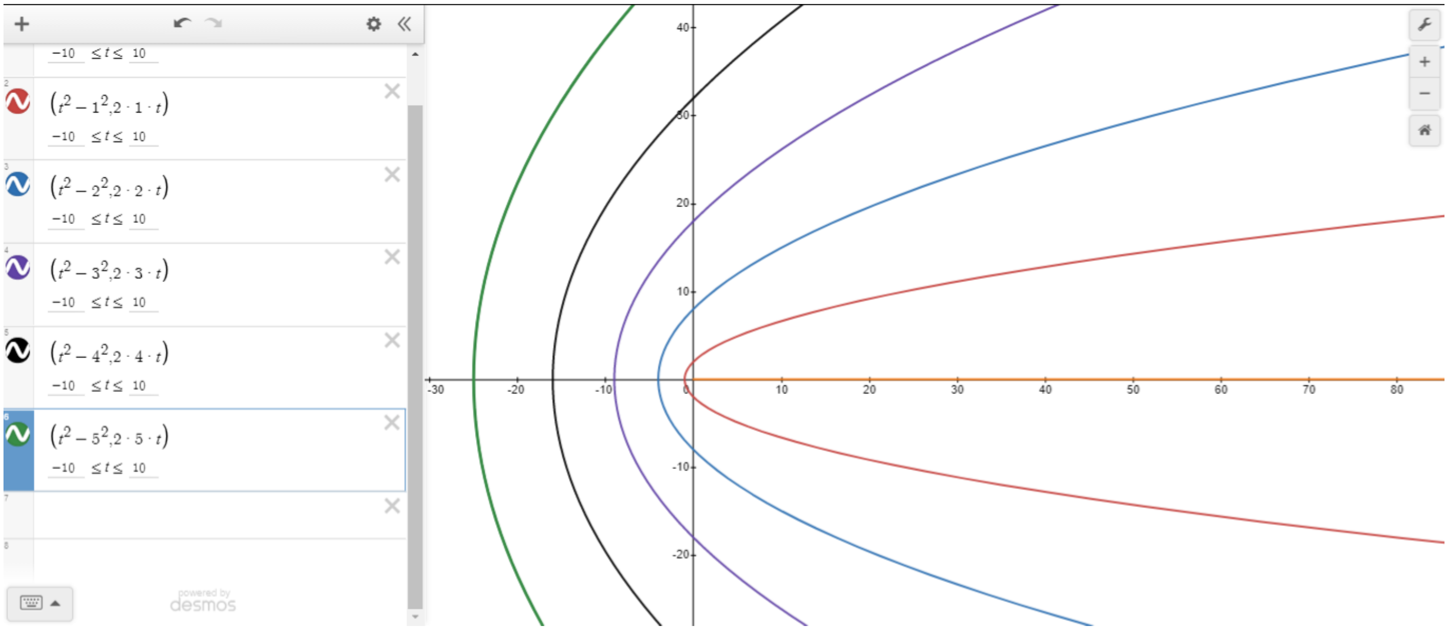


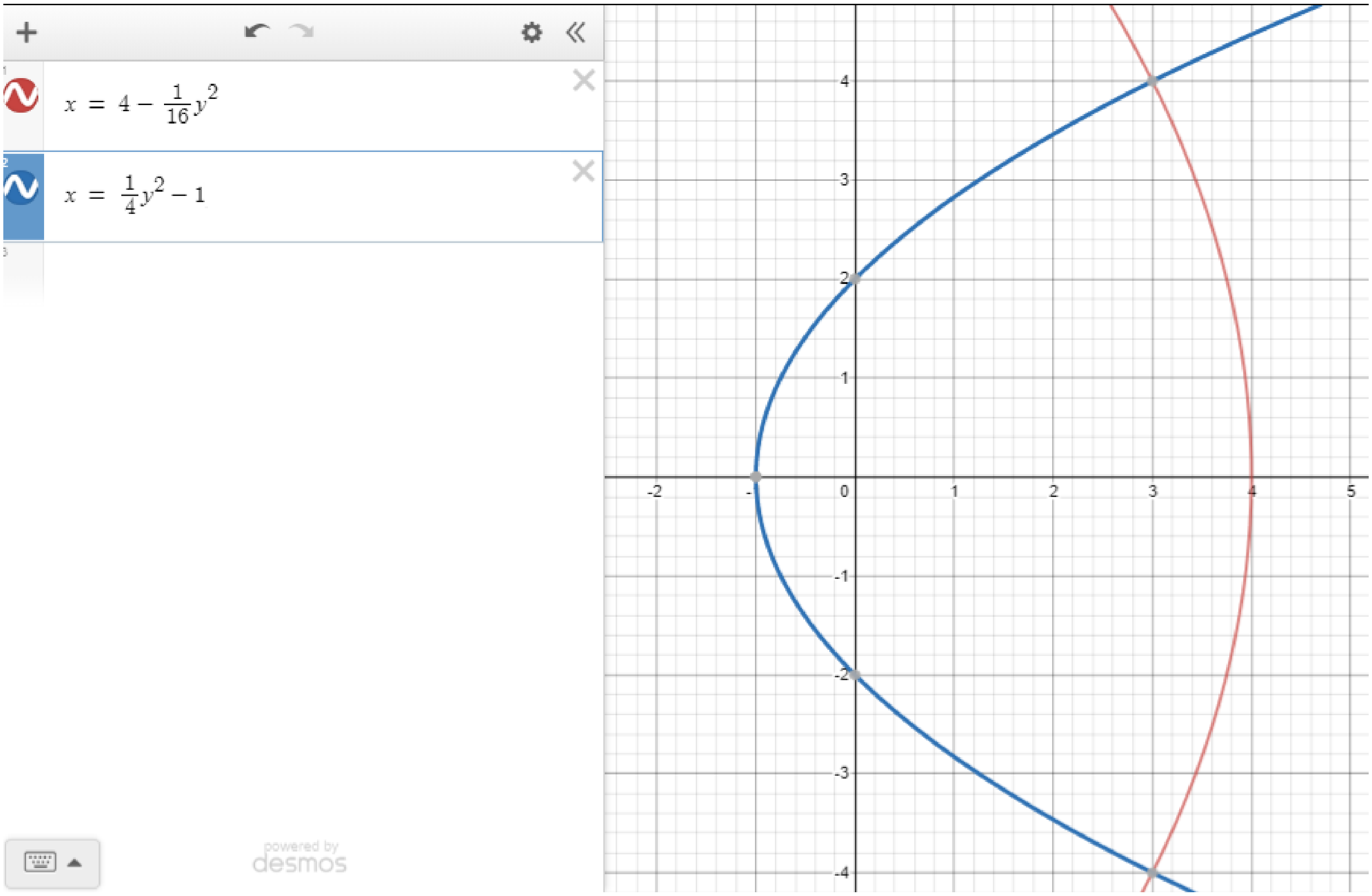
Figure 12: The parabolas in the xy-plane corresponding to horizontal lines in the uv-plane under the transformation T.

Solution to part c: Since $x = u^2 - v^2$ and $y = 2uv$, we see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 2u \cdot 2u - (-2v) \cdot 2v = \boxed{4u^2 + 4v^2}. \tag{38}$$

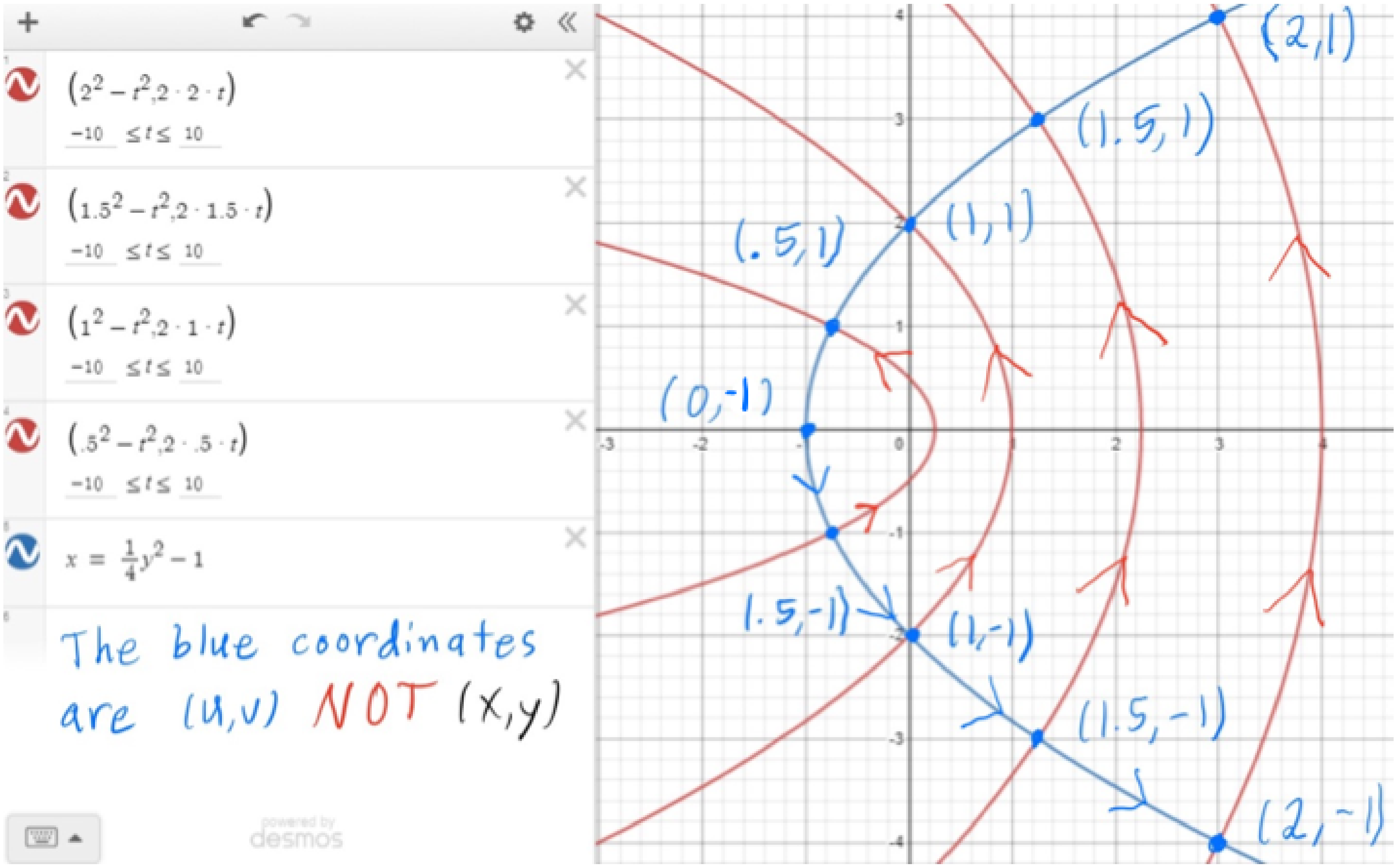
We also observe that $|J(u,v)| = J(u,v) = 4u^2 + 4v^2$ since squares are always bigger than or equal to 0.

d. Use a change of variables into parabolic coordinates to find the area of the region R in the xy-plane bounded by the curves $x = 4 - \frac{1}{16}y^2$ and $x = \frac{1}{4}y^2 - 1$. Sketch a picture of the new region of integration as well.



Solution to part d: We begin by using parts **a** and **b** to see that the parabola $x = 4 - \frac{1}{16}y^2 = 2^2 - \frac{1}{4 \cdot 2^2}y^2$ in the xy-plane is the image under T of the line $u = 2$ (or $u = -2$) in the uv-plane and the parabola $x = \frac{1}{4}y^2 - 1 = \frac{1}{4 \cdot 1^2}y^2 - 1^2$ in the xy-plane is the image under T of the line $v = 1$ (or $v = 1$) in the uv-plane. Note that for a postive number p, $T(2, p)$ is on the upper half of the parabola $x = 4 - \frac{1}{16}y^2$ and $T(2, -p)$ is on the lower half. Similarly, for $T(p, 1)$ is on the upper half of the parabola $x = \frac{1}{4}y^2 - 1$ and $T(-p, 1) = T(p, -1)$ is on the lower half. We also recall that T (almost) bijects the closed right (or left, or upper, or lower) half of the uv-plane to the xy-plane.[4] The picture below puts together all of the previous discussion to show that the region R in the xy-plane is the image under T of the region rectangle $R' = \{(u, v) \mid 0 \leq u \leq 2, -1 \leq v \leq 1\}$ in the uv-plane.

[4]The map T from the uv-plane to the xy-plane is a one-to-one map if you restrict yourself to an open half of the uv-plane and an appropriate closed half of an axis (such as the open left half of the plane and the closed upper half of the y-axis), but T is not one-to-one on the entire uv-plane since $T(a, b) = T(-a, -b)$.

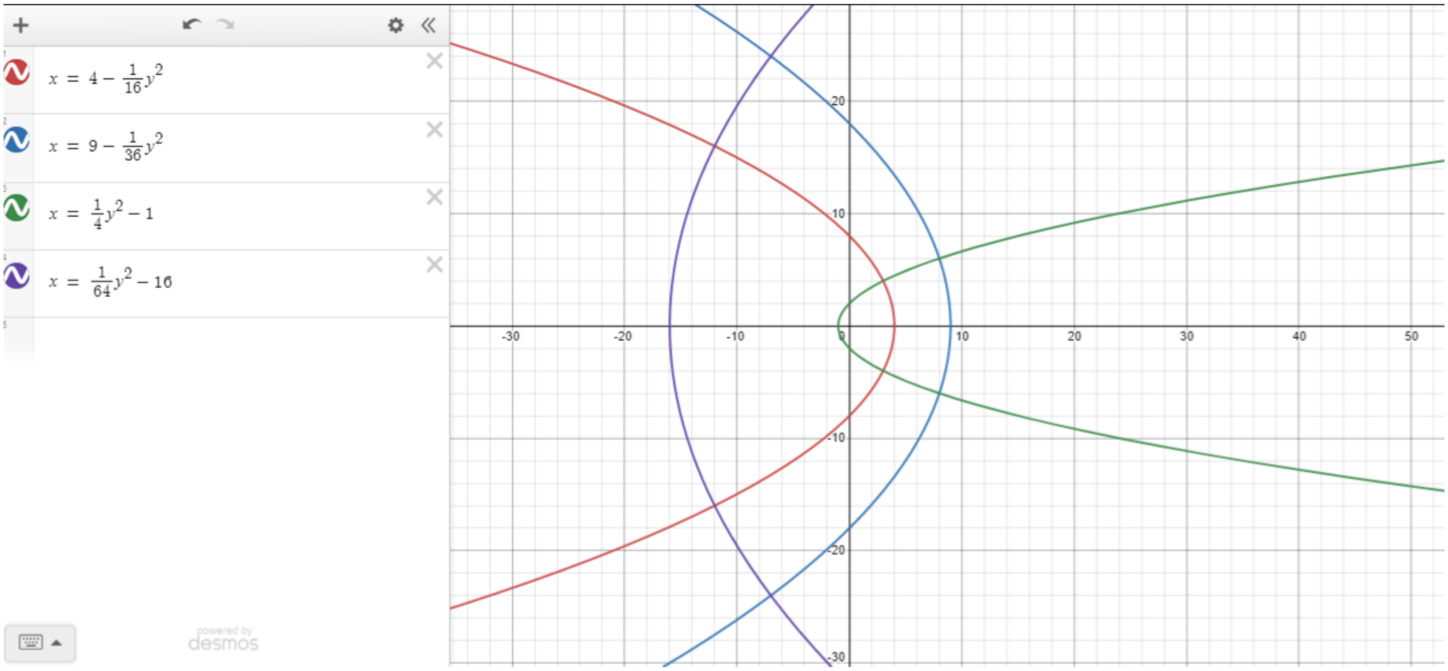


$$\text{Area}(R) = \iint_R 1 dA = \iint_{R'} 1 \cdot |J(u,v)| dA = \int_0^2 \int_{-1}^1 |J(u,v)| dv du \tag{39}$$

$$= \int_0^2 \int_{-1}^1 (4u^2 + 4v^2) dv du = \int_0^2 (4u^2 v + \frac{4}{3}v^3\Big|_{v=-1}^1) du = \int_0^2 (8u^2 + \frac{8}{3}) du \tag{40}$$

$$= \frac{8}{3}u^3 + \frac{8}{3}u\Big|_{u=0}^2 = \boxed{\frac{80}{3}}. \tag{41}$$

e. Use a change of variables into parabolic coordinates to find the area of the curved rectangle R above the x-axis bounded by $x = 4 - \frac{1}{16}y^2, x = 9 - \frac{1}{36}y^2, x = \frac{1}{4}y^2 - 1$, and $x = \frac{1}{64}y^2 - 16$. Sketch a picture of the new region of integration as well.



Solution to part e: We proceed as we did in part **d**. We note that $x = 4 - \frac{1}{16}y^2$ corresponds to $u = 2, -2$, $x = 9 - \frac{1}{36}y^2$ corresponds to $u = 3, -3$, $x = \frac{1}{4}y^2 - 1$ corresponds to $v = 1, -1$, and $x = \frac{1}{64}y^2 - 16$ corresponds to $v = 4, -4$. Since $y = 2uv$ is positive when u and v are both positive (or both negative), we obtain the parabolic rectangle above the x-axis as the image of the region $R' = \{(u, v) \mid 2 \leq u \leq 3, 1 \leq v \leq 4\}$ under T.

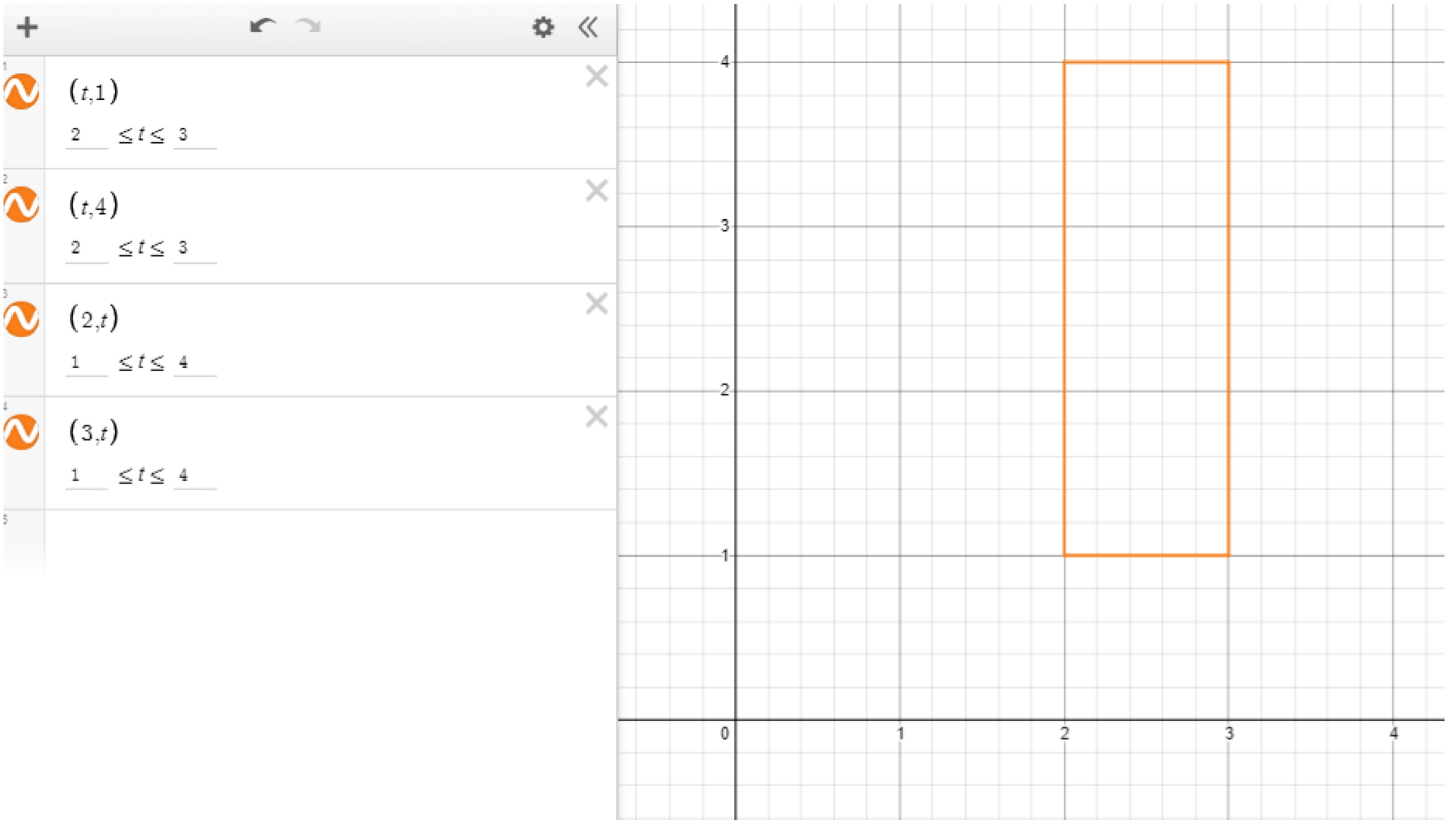


Figure 13: The new region of integration R' in the uv-plane.

We now see that

$$\text{Area}(R) = \iint_R 1dA = \iint_{R'} 1 \cdot |J(u,v)|dA = \int_2^3 \int_1^4 (4u^2 + 4v^2)dvdu \tag{42}$$

$$= \int_2^3 (4u^2v + \frac{4}{3}v^3\Big|_{v=1}^4)du = \int_2^3 (12u^2 + 84)du = 4u^3 + 84u\Big|_2^3 = \boxed{160}. \tag{43}$$

f. Describe the effect of the transformation $(u, v) \mapsto (2uv, u^2 - v^2)$ on horizontal and vertical lines in the uv-plane.[5]

Solution to part f: Let $S(u, v) = (2uv, u^2 - v^2)$. If P is a parabola that opens in the negative x-direction and has its vertex on the positive x-axis, then upon reflection over the line $x = y$, we obtain a parabola P' that opens in the negative y-direction and has its vertex on the positive y-axis. It follows that the image of the vertical line $u = a$ in the uv-plane under the transformation S gives a parabola in the xy-plane that opens in the negative y-direction and has its vertex on the positive y-axis. Similarly, if P is a parabola that opens in the positive x-direction and has its vertex on the negative x-axis, then upon reflection over the line $x = y$, we obtain a parabola P' that opens in the positive y-direction and has its vertex on the negative y-axis. It follows that the image of the horizontal line $v = b$ in the uv-plane under the transformation S gives a parabola in the xy-plane that opens in the positive y-direction and has its vertex on the negative y-axis.

[5] Remember that the transformation $(x, y) \mapsto (y, x)$ reflects points in the xy-plane across the line $y = x$. It will also help to use the results of parts **a.** and **b.** of this problem.

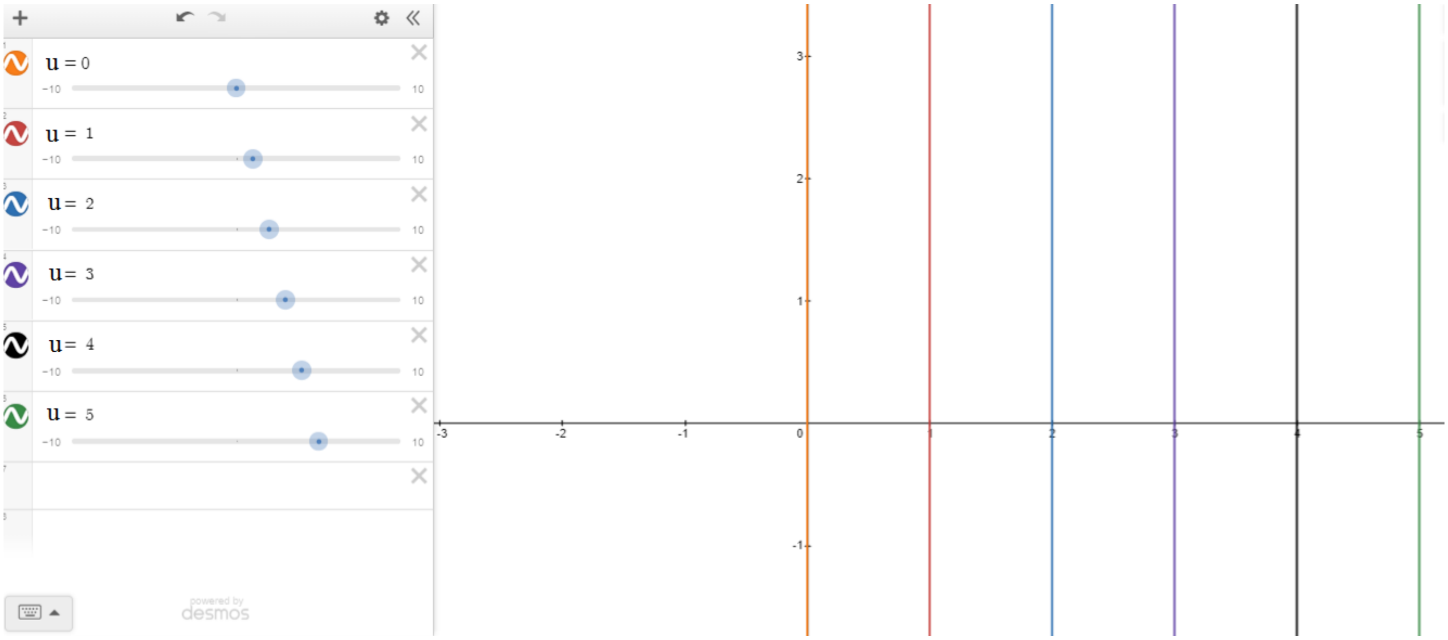


Figure 14: Vertical lines in the uv-plane.

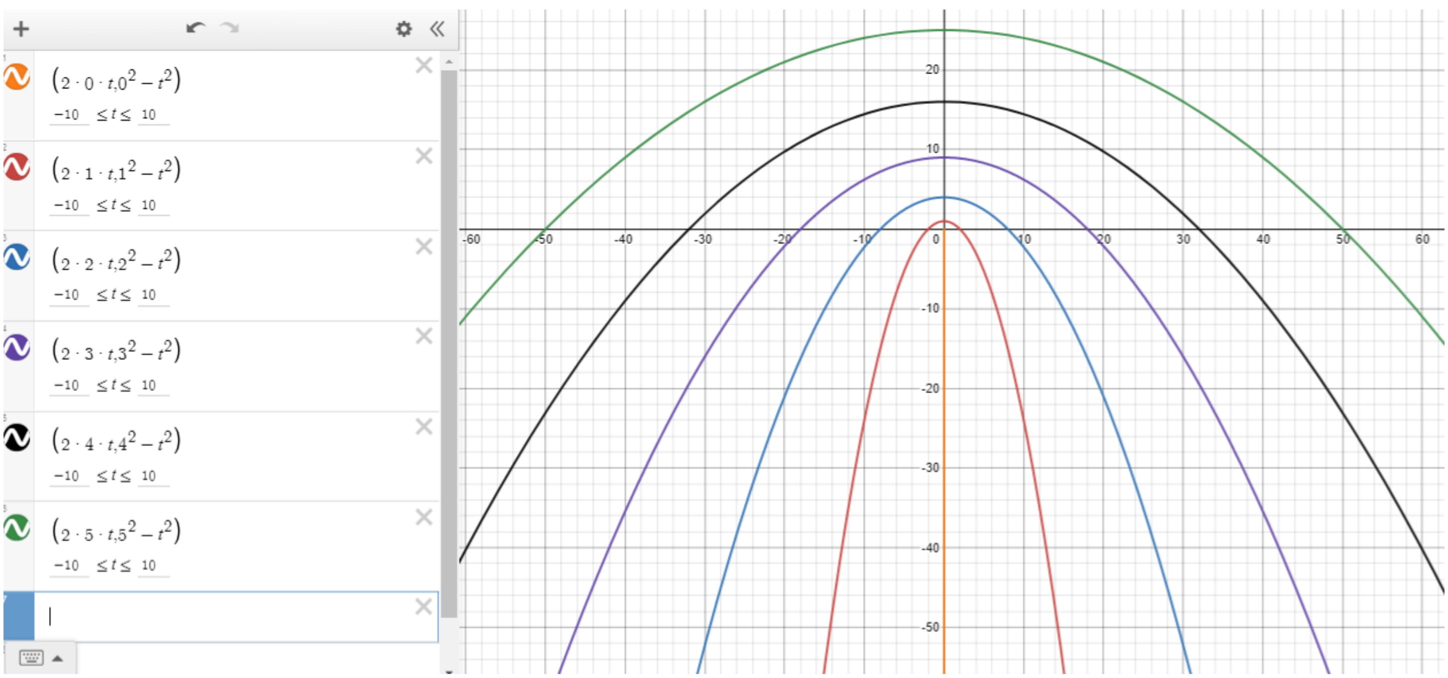


Figure 15: The parabolas in the xy-plane corresponding to vertical lines in the uv-plane under the transformation S.

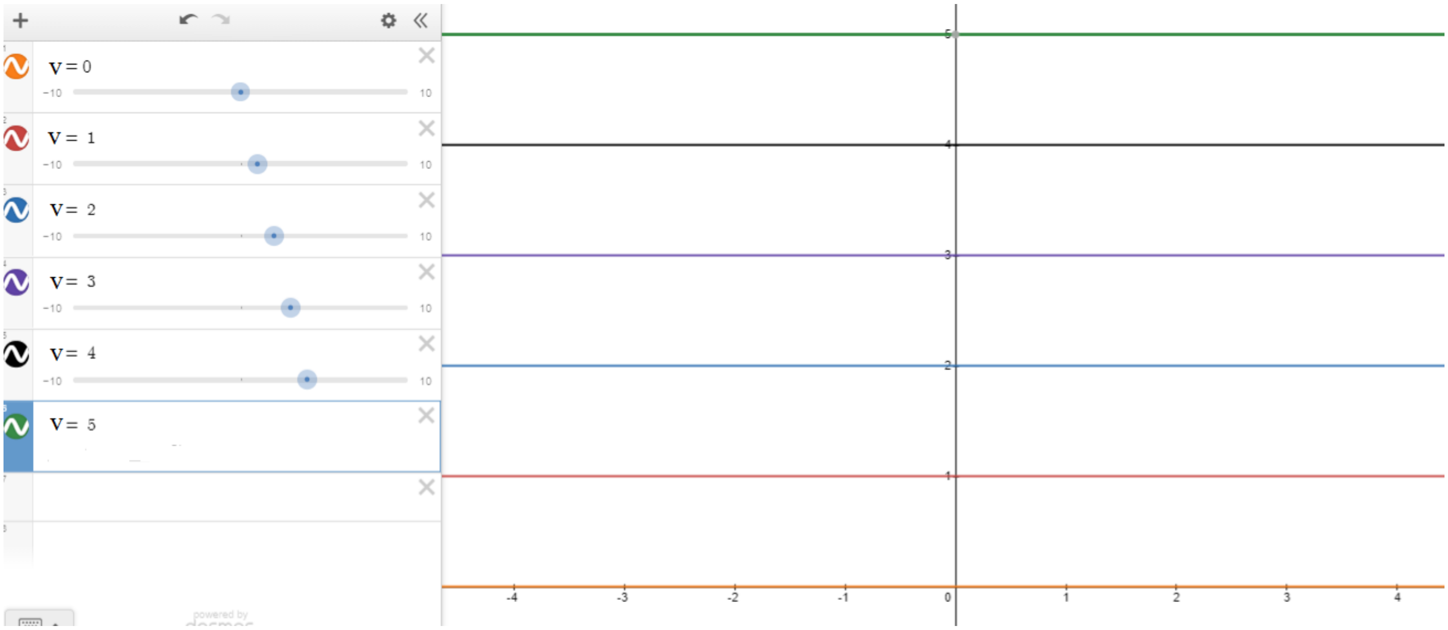


Figure 16: Horizontal lines in the uv-plane.

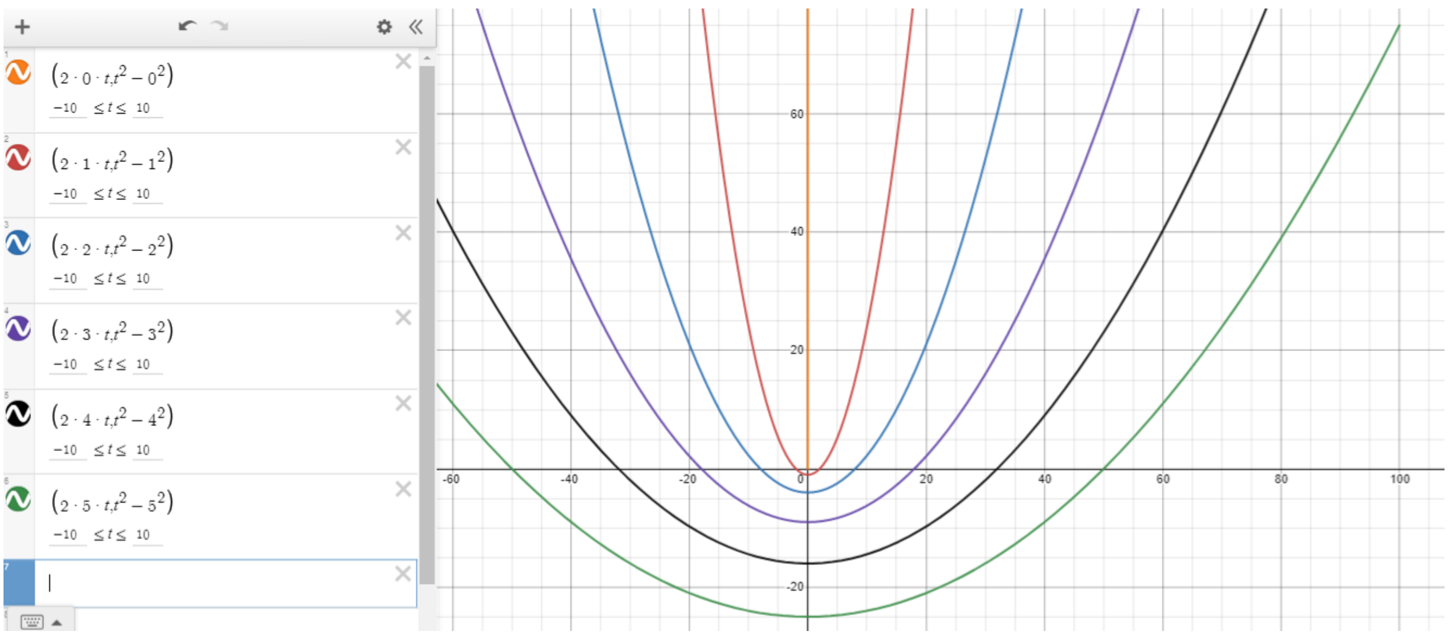


Figure 17: The parabolas in the xy-plane corresponding to horizontal lines in the uv-plane under the transformation S.

g. Show that the parabolas that are the images of the lines $u = a$ and $v = b$ under $T(u, v) = (u^2 - v^2, 2uv)$ are orthogonal to eachother.

Solution to part g: We have already seen in parts **a** and **b** that $T(a, v)$ is the parabola $x = a^2 - \frac{1}{4a^2}y^2$ and $T(u, b)$ is the parabola $x = \frac{1}{4b^2}y^2 - b^2$. We will first find the intersection points of these 2 parabolas, then we will calculate the slope of the tangent lines at the intersection points in order to see that the tangent lines (and hence the curves) are orthogonal.

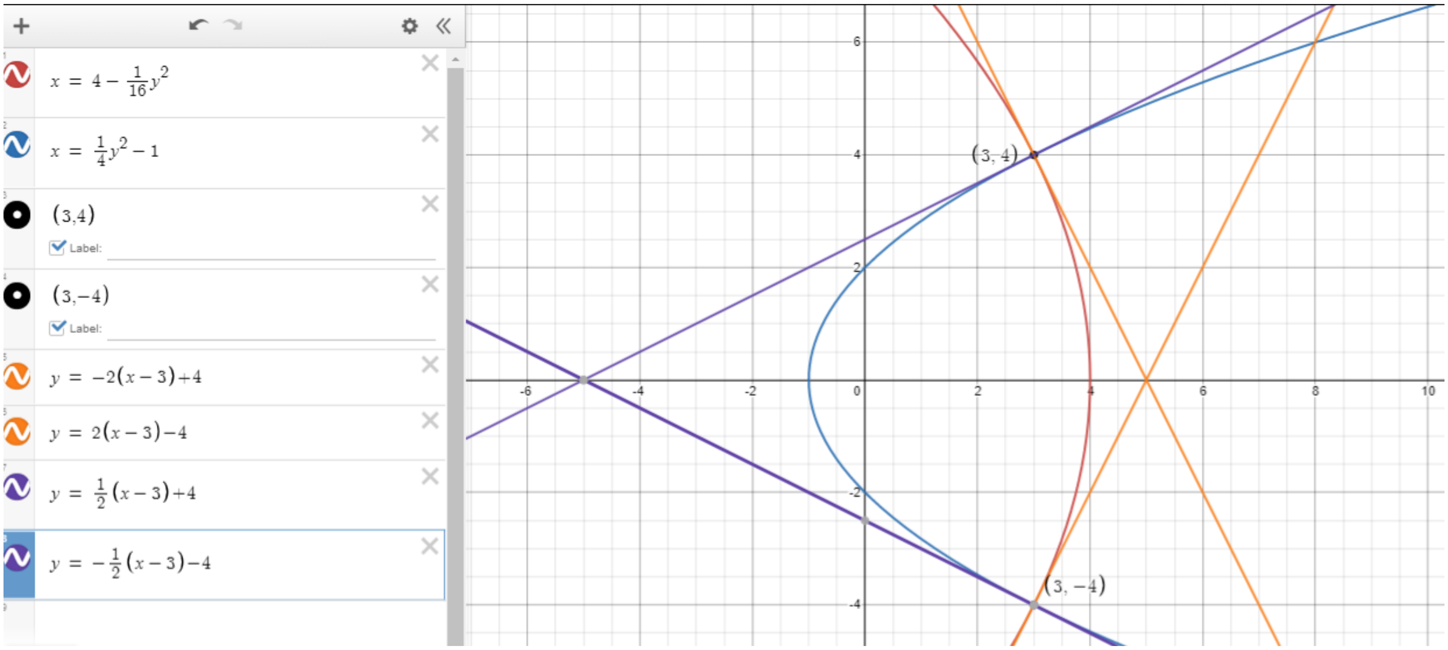


Figure 18: A picture of $T(2, v), T(u, 1)$, and the tangent lines to both curves at their intersection points.

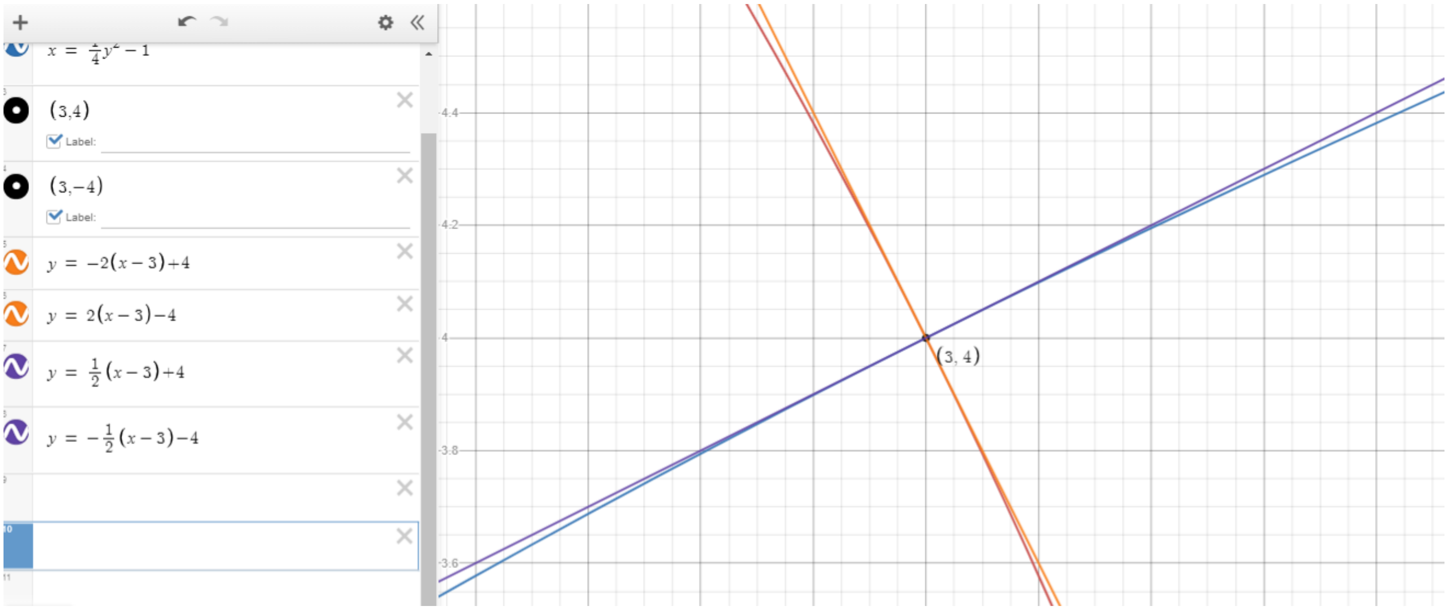


Figure 19: A zoomed in shot around the intersection point (3, 4) to show that the tangent lines (and hence the curves) are perpendicular.

To this end, we see that

$$\begin{matrix} x & = & a^2 - \frac{1}{4a^2}y^2 \\ x & = & \frac{1}{4b^2}y^2 - b^2 \end{matrix} \rightarrow a^2 - \frac{1}{4a^2}y^2 = \frac{1}{4b^2}y^2 - b^2 \rightarrow a^2 + b^2 = (\frac{1}{4a^2} + \frac{1}{4b^2})y^2 \quad (44)$$

$$\rightarrow y^2 = \frac{a^2 + b^2}{\frac{1}{4a^2} + \frac{1}{4b^2}} = 4a^2b^2 \rightarrow y = \pm 2ab \rightarrow x = a^2 - b^2. \quad (45)$$

It follows that $T(a, b) = T(-a, -b) = (a^2 - b^2, 2ab)$ and $T(a, -b) = T(-a, b) = (a^2 - b^2, -2ab)$ are the intersection points of the 2 parabolas. Noting that

$$x = a^2 - \frac{1}{4a^2}y^2 \rightarrow dx = -\frac{1}{2a^2}ydy \rightarrow \frac{dy}{dx} = -2\frac{a^2}{y}, \text{ and} \quad (46)$$

$$x = \frac{1}{4b^2}y^2 - b^2 \rightarrow dx = \frac{1}{2b^2}ydy \rightarrow \frac{dy}{dx} = 2\frac{b^2}{y}, \quad (47)$$

We see that at the point $(a^2 - b^2, 2ab)$, the tangent line to the curve $x = a^2 - \frac{1}{4a^2}y^2$ has a slope of $-\frac{a}{b}$ and the tangent line to the curve $x = \frac{1}{4b^2}y^2 - b^2$ has a slope of $\frac{b}{a}$. Since $-\frac{a}{b} \cdot \frac{b}{a} = -1$, we see that the tangent lines at the point $(a^2 - b^2, 2ab)$ are indeed orthogonal to each other. Similarly, we see that at the point $(a^2 - b^2, -2ab)$, the tangent line to the curve $x = a^2 - \frac{1}{4a^2}y^2$ has a slope of $\frac{a}{b}$ and the tangent line to the curve $x = \frac{1}{4b^2}y^2 - b^2$ has a slope of $-\frac{b}{a}$. Since $\frac{a}{b} \cdot (-\frac{b}{a}) = -1$, we see that the tangent lines at the point $(a^2 - b^2, 2ab)$ are indeed orthogonal to each other.

Remark: We see that the parabolas produced by S in part **f** also share this orthogonality property since orthogonality is preserved under reflections.