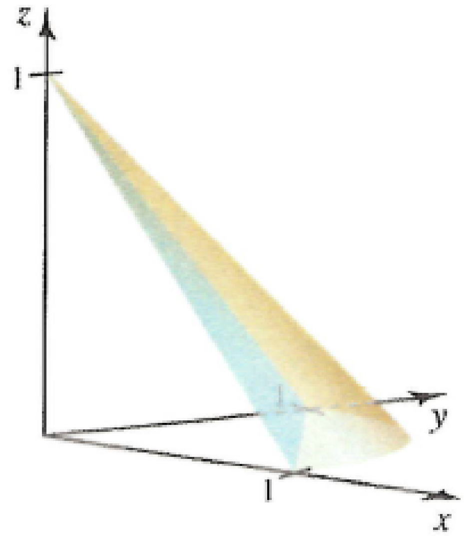
Problem 1: Find the volume of the solid S in the first octant that is bounded by the cone $z = 1 - \sqrt{x^2+y^2}$ and the plane $x + y + z = 1$.



Solution 1: We see that

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz \tag{1}$$

$$= \int_0^1 \int_0^{1-z} x\Big|_{1-z-y}^{\sqrt{(1-z)^2-y^2}} dydz \tag{2}$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2-y^2} - (1-z-y)\right) dydz. \tag{3}$$

We see that evaluating (the difficult part of) the inner integral in (3) is tantamount to evaluating

$$\int \sqrt{1-y^2}dy, \tag{4}$$

which is certainly possible, but it is difficult and computationally intensive, so we will evaluate the volume by an alternative method. If we more closely examine the integrals in (1), then we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy \tag{5}$$

calculates the area of the cross section C_z shown in figure 1.

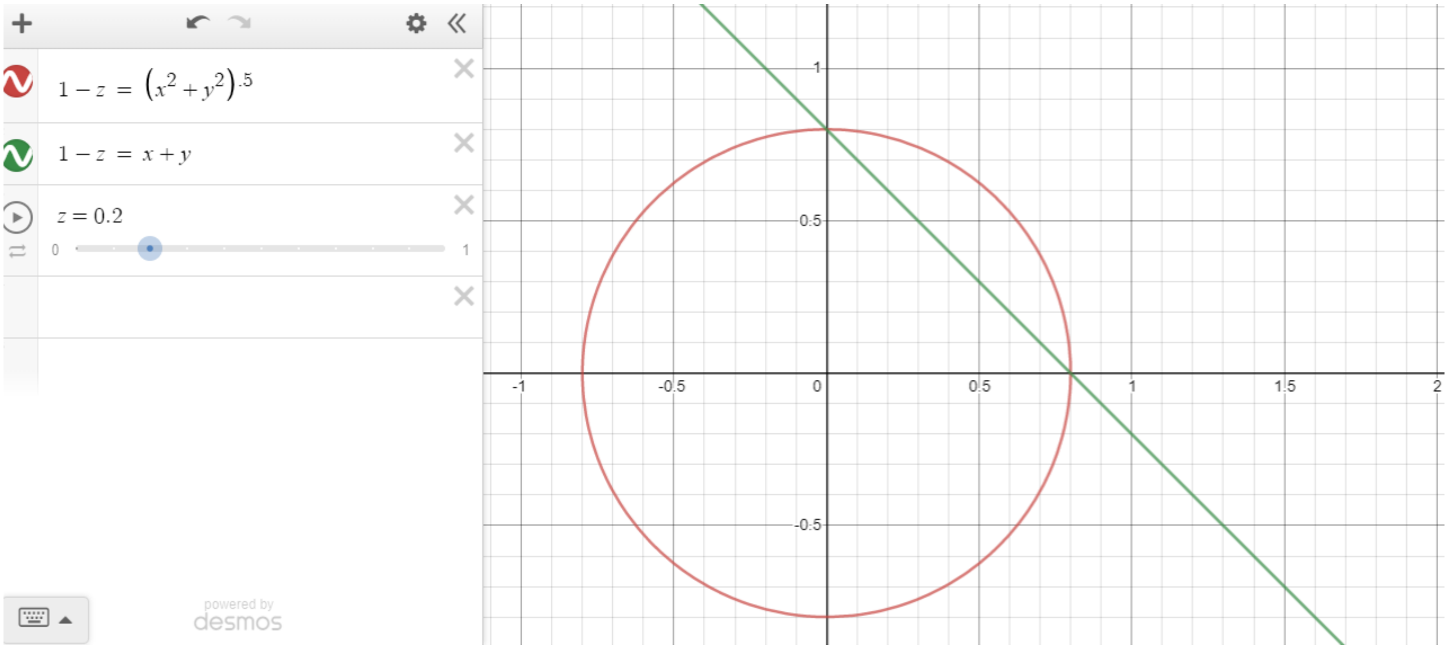


Figure 1: The cross section of S at a particular height z.

Using elementary Euclidena geometry, we see that

$$\int_0^{1-z}\int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy = \text{Area}(C_z)$$
$$= \frac{1}{4}\pi(1-z)^2 - \frac{1}{2}(1-z)^2 = \frac{\pi-2}{4}(1-z)^2. \quad (6)$$

It follows that

$$\int_0^1\int_0^{1-z}\int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz = \int_0^1 \frac{\pi-2}{4}(1-z)^2dz = -\frac{\pi-2}{12}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi-2}{12}}. \quad (7)$$

Solution 2: Let C be the portion of the cone $z = 1-\sqrt{x^2+y^2}$ that is in the first quadrant and let T be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. We see that S is simply the solid C with the solid T removed from it. Recalling that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2h$, and that the volume of a tetrahedron with height h and a base of area b is $\frac{1}{3}bh$, we see that

$$\text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T) = \frac{1}{3}\pi\cdot 1^2\cdot 1\underbrace{\cdot\frac{1}{4}}_{\text{QI}} - \frac{1}{3}\cdot\underbrace{(\frac{1}{2}\cdot 1\cdot 1)}_{\text{Area of base}}\cdot 1 = \boxed{\frac{\pi-2}{12}}. \quad (8)$$

Solution 3: We proceed as we did in Solution 2, but we will now derive the formula for the volume of C and T by using a triple integral in cylindrical coordinates for C and a triple integral in Cartesian coordinates for T. Recalling that the Cartesian equation $z = 1 - \sqrt{x^2 + y^2}$ is rewritten as $z = 1 - r$ in cylindrical coordinates, we see that

$$\text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T) \tag{9}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1-r} r dz dr d\theta - \int_0^1 \int_0^{1-z} \int_0^{1-z-y} dx dy dz \tag{10}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 rz\Big|_0^{1-r} dr d\theta - \int_0^1 \int_0^{1-z} x\Big|_0^{1-z-y} dy dz \tag{11}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (r - r^2) dr d\theta - \int_0^1 \int_0^{1-z} (1 - z - y) dy dz \tag{12}$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2}r^2 - \frac{1}{3}r^2\Big|_0^1) d\theta - \int_0^1 ((1-z)y - \frac{1}{2}y^2\Big|_{y=0}^{1-z}) dy dz \tag{13}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{6} d\theta - \int_0^1 \frac{1}{2}(1-z)^2 dz \tag{14}$$

$$= \frac{1}{6}\theta\Big|_0^{\frac{\pi}{2}} - \frac{1}{6}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi - 2}{12}}. \tag{15}$$

Problem 2: Evaluate

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz. \tag{16}$$

Hint: A different order of integration can make the problem easier, even though it is not necessary.

Solution: We see that trying to evaluate the inner integral in the current order of integration is tantamount to evaluating

$$\int c_1 \sin(c_2\sqrt{y})dy, \tag{17}$$

which is very difficult, so we decide to change the order of integration in hopes that the inner integral becomes easier to evaluate. We see that integrating with respect to z in the inner integral is not any easier since z and y are symmetric in the integrand, so we decide to integrate with respect to x in the inner integral in our new order of integration. Since z and y are symmetric in the integrand, the difficulty of the integrations doesn't seem to change if we use $dxdydz$ or $dxdzdy$, so we will use the order $dxdydz$ in order to reduce our workload by only changing the order of dx and dy instead of changing the order of dx, dy, and dz. We see that the bounds that we have in (**??**) tell us that

$$\begin{array}{ccccc} 1 & \leq & z & \leq & 4 \\ z & \leq & x & \leq & 4z \\ 0 & \leq & y & \leq & \pi^2 \end{array} \rightarrow \begin{array}{ccccc} 1 & \leq & z & \leq & 4 \\ 0 & \leq & y & \leq & \pi^2 \\ z & \leq & x & \leq & 4z \end{array}. \tag{18}$$

Thankfully, we didn't have to do any work to interchange the order of dx and dy since the bounds for y in the first order of integration were independent of x. We now see that

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz = \int_1^4 \int_0^{\pi^2} \int_z^{4z} \sin(\sqrt{yz})x^{-\frac{3}{2}}dxdydz \tag{19}$$

$$= \int_1^4 \int_0^{\pi^2} -2\sin(\sqrt{yz})x^{-\frac{1}{2}}\Big|_{x=z}^{4z} dydz \tag{20}$$

$$= \int_1^4 \int_0^{\pi^2} \left(-2\sin(\sqrt{yz})(4z)^{-\frac{1}{2}} + 2\sin(\sqrt{yz})z^{-\frac{1}{2}}\right) dydz \tag{21}$$

$$= \int_1^4 \int_0^{\pi^2} \left(-\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} + 2\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}\right) dydz = \int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz. \tag{22}$$

We see that evaluating the inner integral at the end of (22) is again tantamount to evaluating the integral in (17), so we decide to change the order of integration once again. Note that this is equivalent to having decided to use the order $dxdzdy$ from the beginning, but we were not able to see that $dxdzdy$ was the best order of integration until now. Nonetheless, our initial change in the order of integration did allow us to make progress despite not being the best possible order of integration.

$$\int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz = \int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy. \tag{23}$$

Recalling that y does not change when evaluating the inner integral with respect to z, we treat y as a constant (relative to z) to perform the u-substituion

$$u = \sqrt{yz}, du = \frac{\sqrt{y}}{2\sqrt{z}} dz, dz = \frac{2\sqrt{z}}{\sqrt{y}} du. \tag{24}$$

We now see that

$$\int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy = \int_0^{\pi^2} \int_{z=1}^4 \frac{2\sin(u)}{\sqrt{y}} dudy \tag{25}$$

. . . .

$$= \int_0^{\pi^2} \frac{-2\cos(u)}{\sqrt{y}}\bigg|_{z=1}^4 dy = \int_0^{\pi^2} \frac{-2\cos(\sqrt{yz})}{\sqrt{y}}\bigg|_{z=1}^4 dy \tag{26}$$

. . . .

$$= \int_0^{\pi^2} \left(\frac{-2\cos(\sqrt{4y})}{\sqrt{y}} + \frac{2\cos(\sqrt{y})}{\sqrt{y}}\right) dy \tag{27}$$

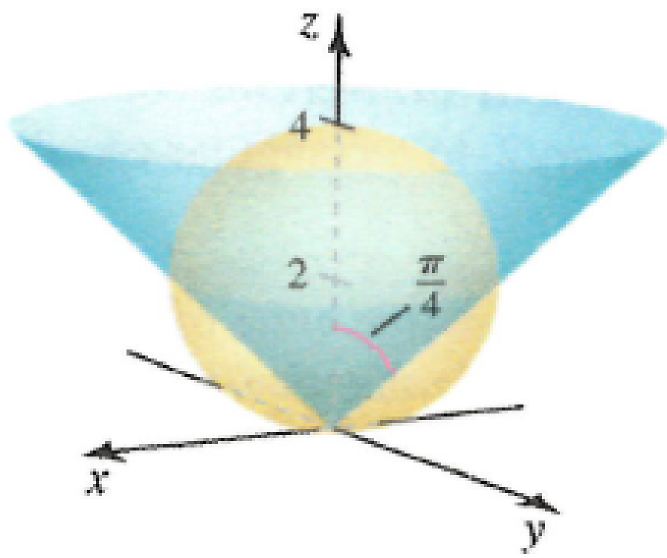
. . . .

$$\overset{u=\sqrt{y}}{=} \int_{y=0}^{\pi^2} (-4\cos(2u) + 4\cos(u))\, du = (-2\sin(2u) + 4\sin(u))\bigg|_{y=0}^{\pi^2} \tag{28}$$

. . . .

$$= (-2\sin(2\sqrt{y}) + 4\sin(\sqrt{y}))\bigg|_{y=0}^{\pi^2} = \boxed{0}. \tag{29}$$

Problem 3: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.



First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. Remember that the angle φ is measured from the z-axis and satisfies $0 \leq \varphi \leq \pi$, not $0 \leq \varphi \leq 2\pi$. Also remember that this cross section is showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

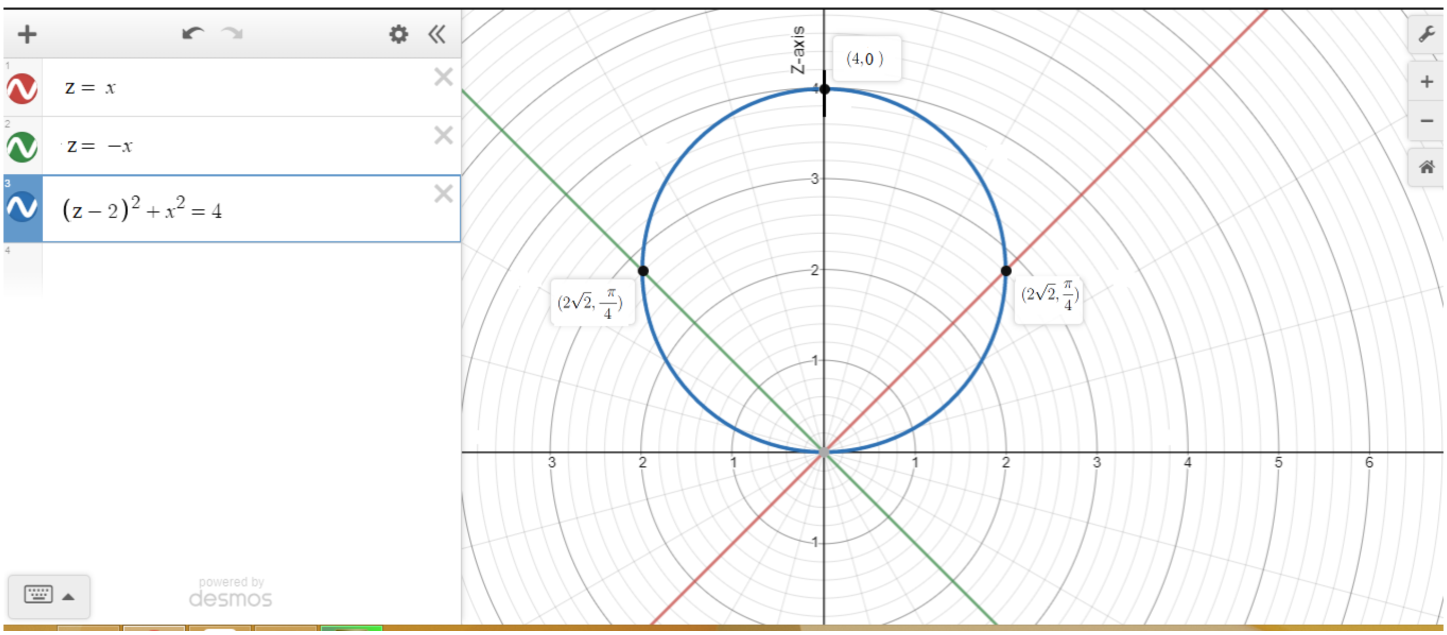


Figure 2: The xz-plane cross section in spherical coordinates.

We now see that for any $\theta \in [0, 2\pi)$ we have that $\frac{\pi}{4} \le \varphi \le \frac{\pi}{2}$. Recalling that the blue circle is defined by $\rho = 4\cos(\varphi)$, we see that once φ is also chosen we have that $0 \le \rho \le 4\cos(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{30}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=0}^{4\cos(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\underbrace{\cos^3(\varphi)}_{u^3}\underbrace{\sin(\varphi)d\varphi}_{-du}d\theta \tag{31}$$

$$= -\frac{64}{3}\int_0^{2\pi}\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} u^3 du d\theta = -\frac{64}{3}\int_0^{2\pi} \frac{1}{4}u^4\Big|_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \tag{32}$$

$$= -\frac{64}{3}\int_0^{2\pi} \frac{1}{4}\cos^4(\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{64}{3}\int_0^{2\pi} -\frac{1}{16}d\theta = -\frac{64}{3}\cdot 2\pi\cdot\frac{-1}{16} = \boxed{\frac{8\pi}{3}}. \tag{33}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). Remember that this cross section is also showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

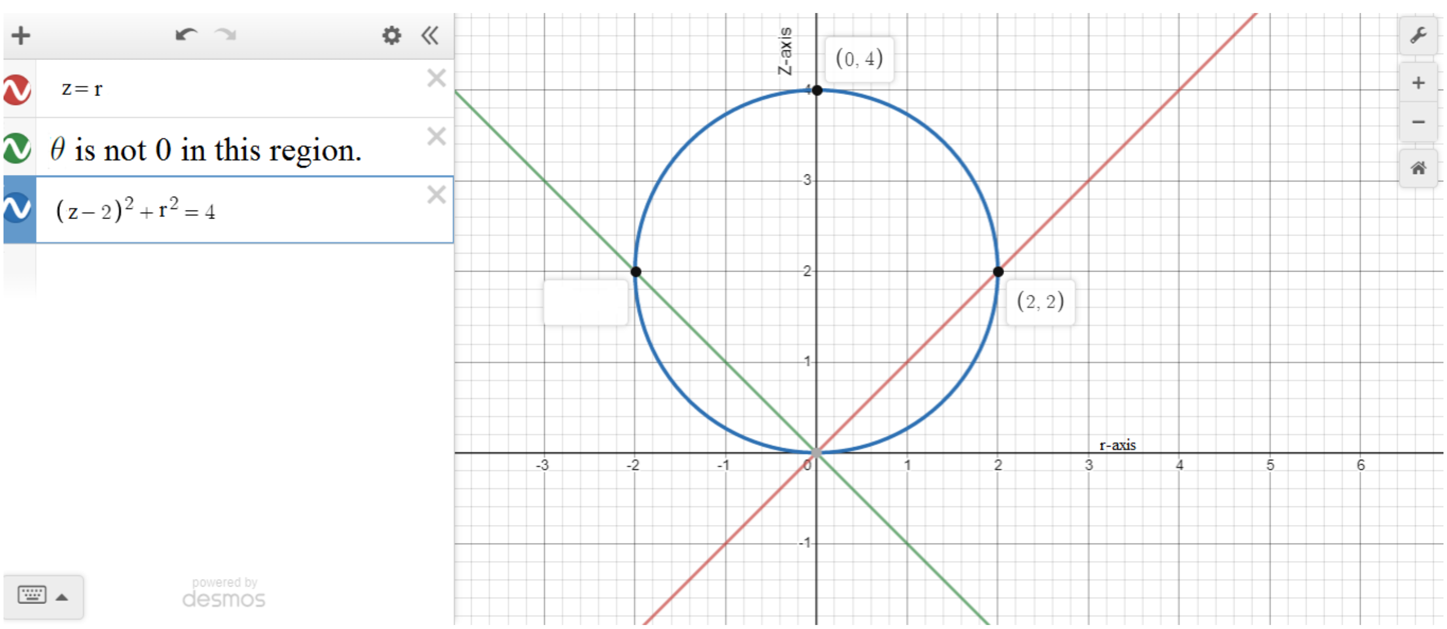


Figure 3: The xz-plane cross section in cylindrical coordinates.

We now see that for any $0 \le \theta < 2\pi$ we have that $0 \le z \le 2$. Noting that we have

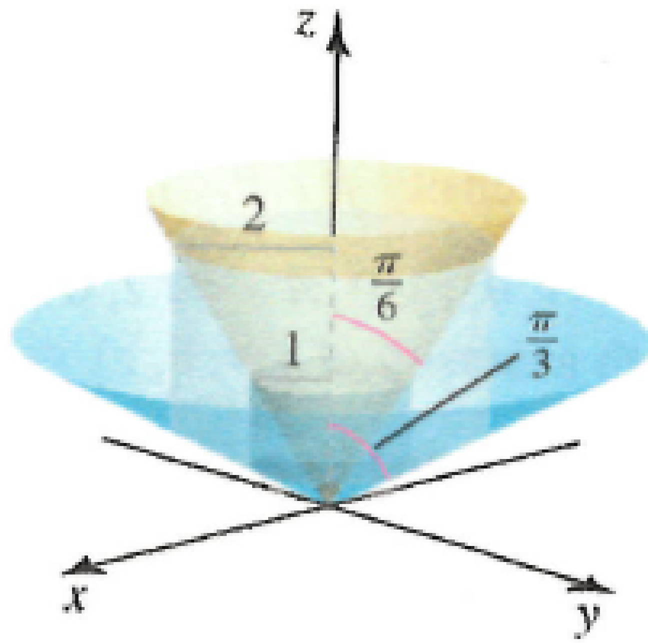
$r = \sqrt{4-(z-2)^2} = \sqrt{4z-z^2}$ on the blue circle, we see that once z is chosen we have $z \le r \le \sqrt{4z-z^2}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_0^2\int_z^{\sqrt{4z-z^2}} rdrdzd\theta \tag{34}$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^2\Big|_z^{\sqrt{4z-z^2}} dzd\theta = \int_0^{2\pi}\int_0^2 (2z-z^2)dzd\theta \tag{35}$$

$$\int_0^{2\pi}(z^2-\frac{1}{3}z^3)\Big|_0^2 d\theta = \int_0^{2\pi}\frac{4}{3}d\theta = \boxed{\frac{8\pi}{3}}. \tag{36}$$

Problem 4: Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.



First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

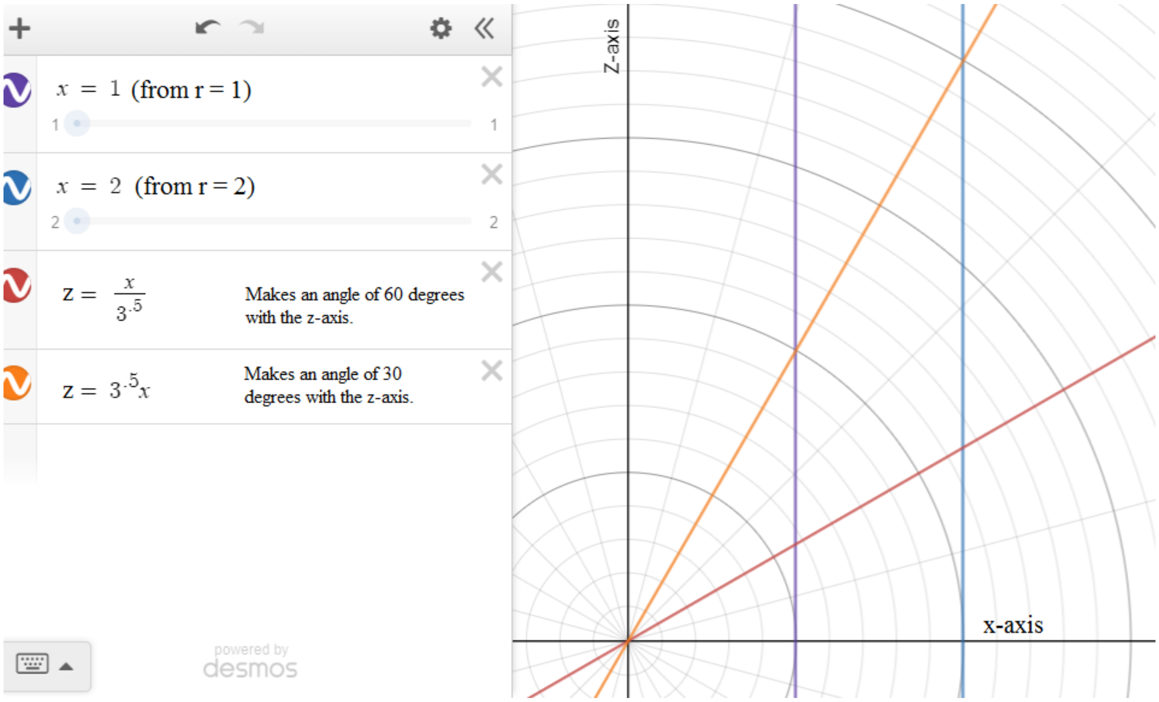


Figure 4: The xz-plane cross section in spherical coordinates.

We see that for any $0 \le \theta < 2\pi$ we have $\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$. Noting that $r = \rho\sin(\varphi)$, we see that when $r = 1$ we have $\rho = \csc(\varphi)$ and when $r = 2$ we have $\rho = 2\csc(\varphi)$. It follows that once φ is also chosen we have $\csc(\varphi) \le \rho \le 2\csc(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{37}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=\csc(\varphi)}^{2\csc(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{7}{3}\csc^2(\varphi)d\varphi d\theta \tag{38}$$

$$= \int_0^{2\pi} -\frac{7}{3}\cot(\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{39}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

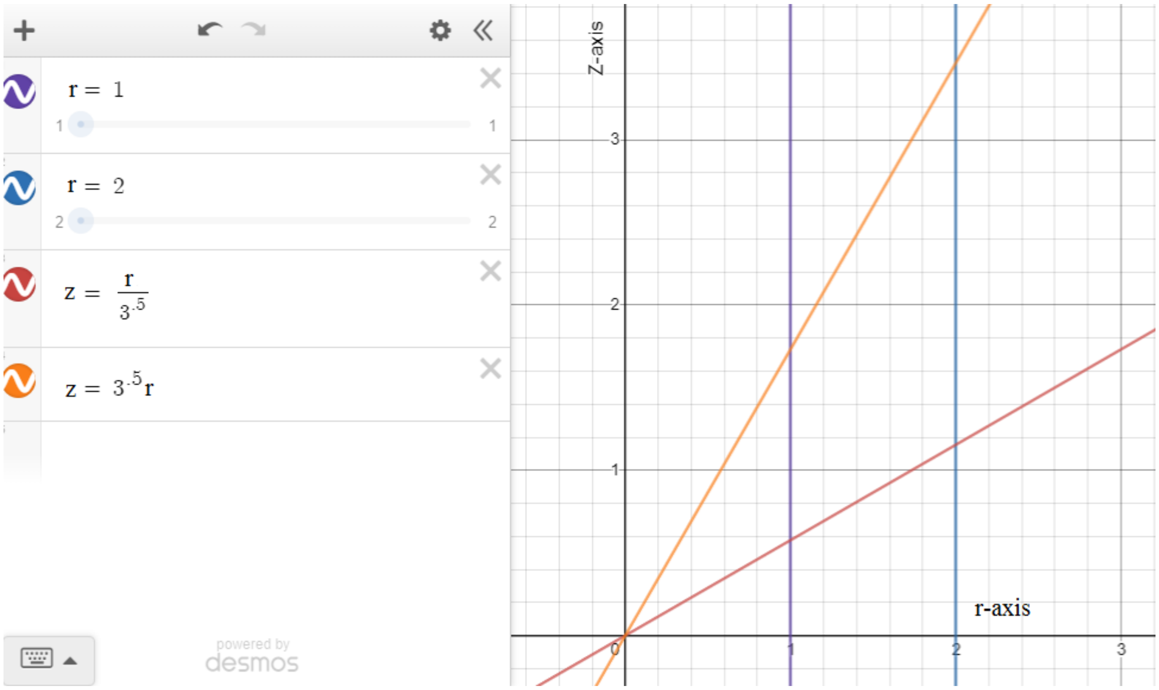


Figure 5: The xz-plane cross section in cylindrical coordinates.

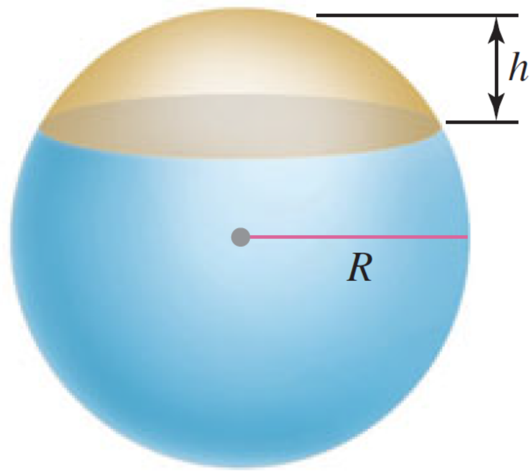
We note that for any $0 \leq \theta < 2\pi$ we have $1 \leq r \leq 2$. Once r is also chosen, we see that $\frac{1}{\sqrt{3}}r \leq z \leq r\sqrt{3}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_1^2\int_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} rdzdrd\theta \tag{40}$$

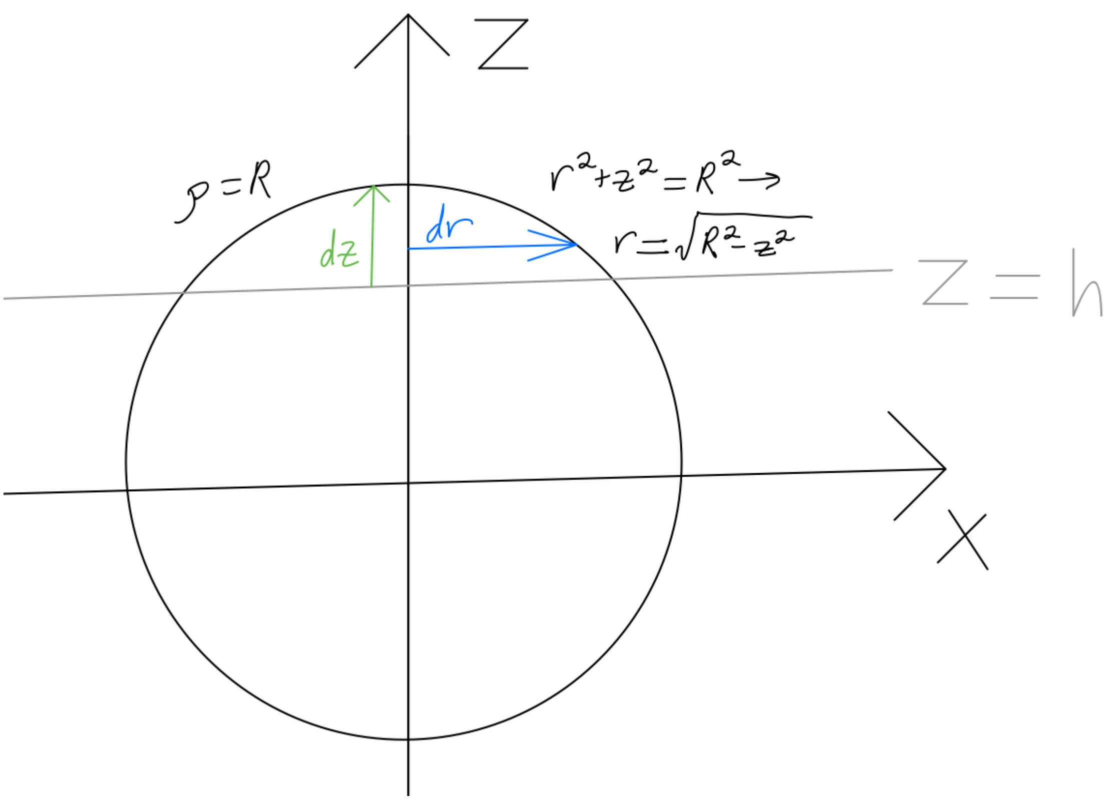
$$= \int_0^{2\pi}\int_1^2 rz\Big|_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}}drd\theta = \int_0^{2\pi}\int_1^2 \frac{2}{\sqrt{3}}r^2drd\theta = \int_0^{2\pi}\frac{2}{3\sqrt{3}}r^3\Big|_1^2 d\theta \tag{41}$$

$$= \int_0^{2\pi}\frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{42}$$

Problem 5: Find the volume of S, the cap of a sphere of radius R with thickness h.



Solution 1: We will first solve this problem using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane, which corresponds to the $\theta = 0$ and $\theta = \pi$ cross sections combined. Since we are working in cylindrical coordinates, the cross section will be handled in coordinates similar to Cartesian coordinates.



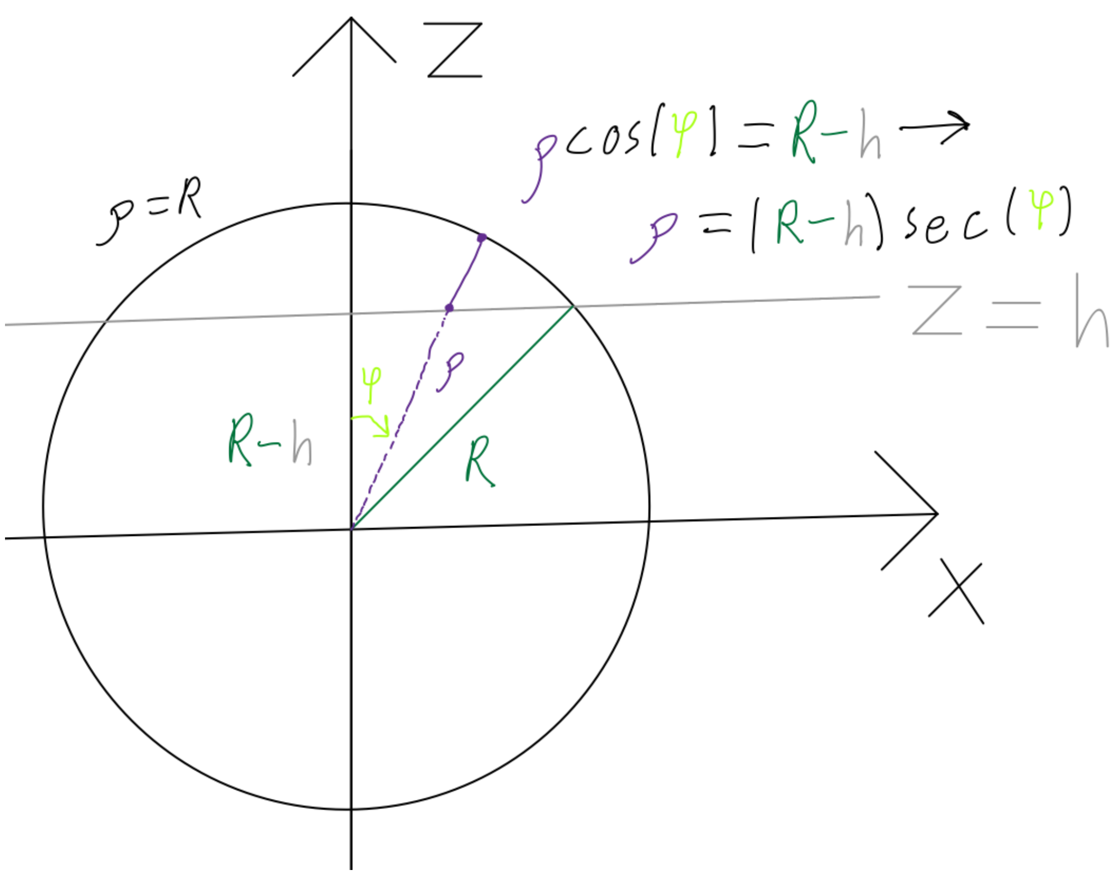
$$\text{Vol}(S) = \int_0^{2\pi}\int_{R-h}^{R}\int_0^{\sqrt{R^2-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_{R-h}^{R} \frac{1}{2}r^2\Big|_{r=0}^{\sqrt{R^2-z^2}} dz\,d\theta \tag{43}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_{R-h}^{R}(R^2-z^2)dz\,d\theta = \frac{1}{2}\int_0^{2\pi}(R^2z-\frac{1}{3}z^3\Big|_{z=R-h}^{R})d\theta \tag{44}$$

$$= \frac{1}{2}\int_0^{2\pi}(R^3-\frac{1}{3}R^3-(R^2(R-h)-\frac{1}{3}(R-h)^3))d\theta = \frac{1}{2}\int_0^{2\pi}(Rh^2-\frac{1}{3}h^3)d\theta \tag{45}$$

$$= \pi(Rh^2-\frac{1}{3}h^3) = \boxed{\frac{\pi}{3}h^2(3R-h)}. \tag{46}$$

Solution 2: We will now solve this problem using spherical coordinates. Due to the symmetry of our solid with respect to θ we once again begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be handled in coordinates similar to polar coordinates.



$$\text{Vol}(S) = \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}\int_{(R-h)\sec\phi}^{R} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{47}$$

$$= \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=(R-h)\sec\phi}^{R} d\varphi d\theta \tag{48}$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi)-(R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi d\theta \tag{49}$$

$$= \frac{2\pi}{3}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi)-(R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi \tag{50}$$

$$= \frac{2\pi}{3}\left(\int_0^{\cos^{-1}(\frac{R-h}{R})}R^3\sin(\varphi)d\varphi - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi\right) \tag{51}$$

$$= \frac{2\pi}{3}\left(-R^3\cos(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})} - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\tan(\varphi)\sec^2(\varphi))d\varphi\right) \tag{52}$$

$$\overset{u=\tan(\varphi)}{=} \frac{2\pi}{3}\left(-R^3(\frac{R-h}{R}) - (-R^3\cdot 1) - \frac{1}{2}(R-h)^3\tan^2(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right) \tag{53}$$

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-\cos^2(\varphi)}{\cos^2(\varphi)}\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right) \tag{54}$$

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-(\frac{R-h}{R})^2}{(\frac{R-h}{R})^2}\right) \tag{55}$$

$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{R^2-(R-h)^2}{(R-h)^2}\right) \tag{56}$$

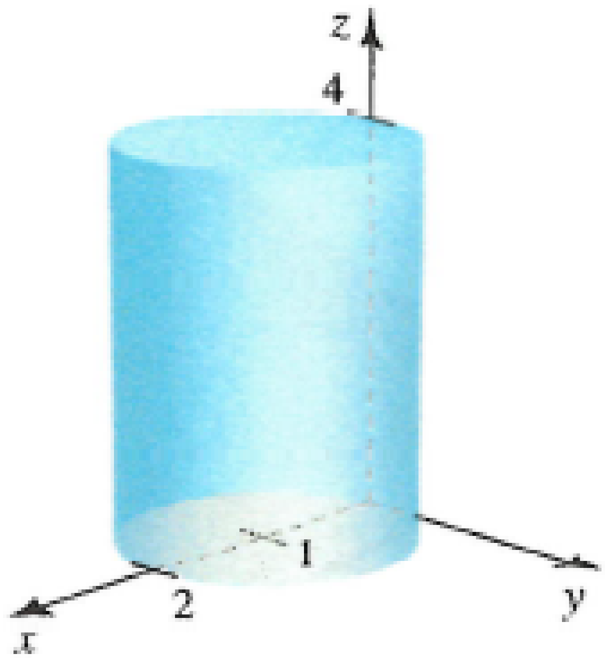
$$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)(2Rh-h^2)\right) \tag{57}$$

. . . .

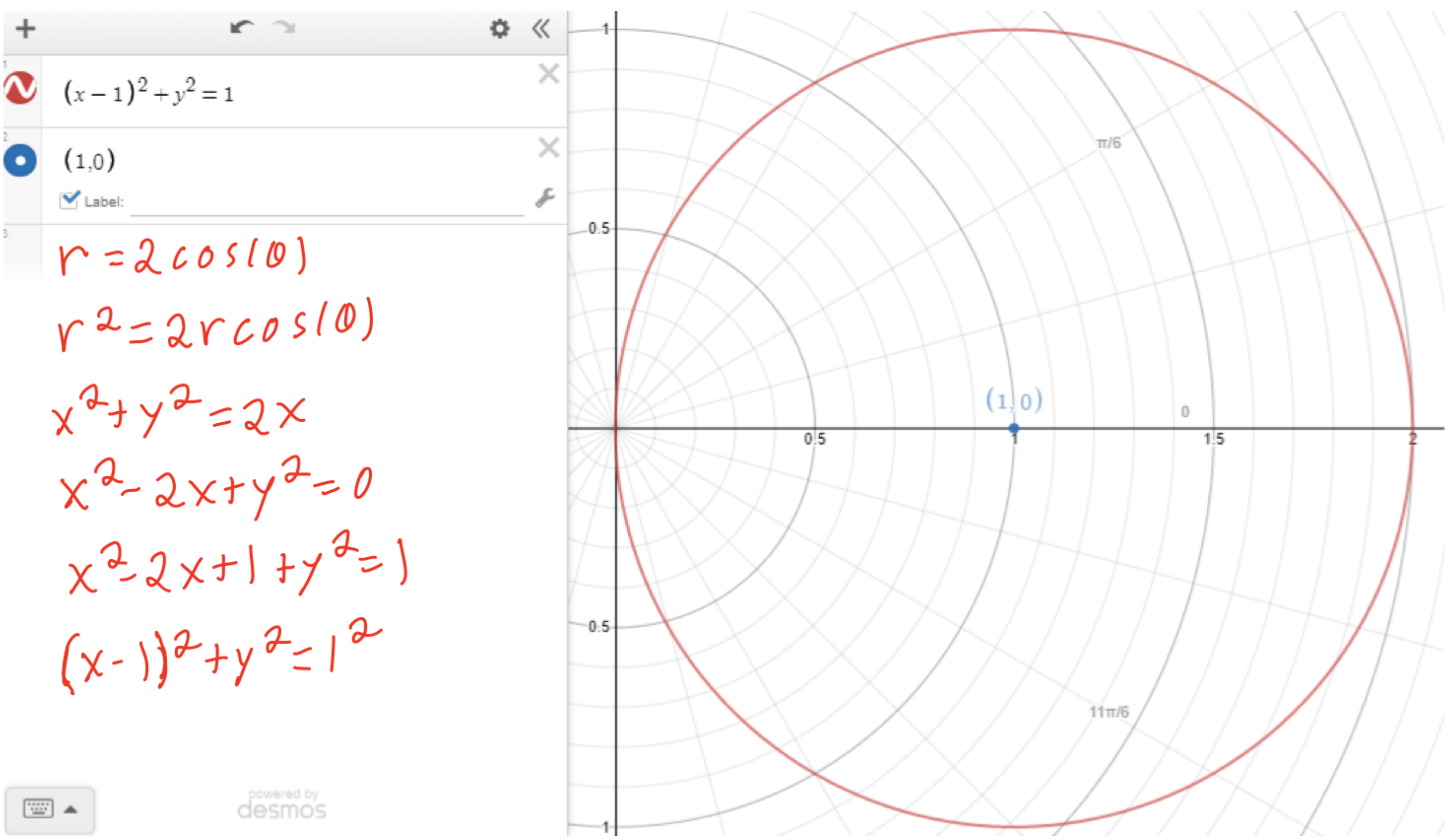
$$= \frac{\pi}{3}(2R^2h - 2R^2h + 2Rh^2 + Rh^2 - h^3) = \boxed{\frac{\pi}{3}h^2(3R - h)}. \tag{58}$$

Remark: In both solutions we can easily check our final answer by noting that $h = 0$ results in a volume of 0, $h = R$ results in a volume of $\frac{2\pi}{3}R^3$ which is indeed the volume of a hemisphere of radius R, and $h = -R$ results in a volume of $\frac{4}{3}R^3$ which is indeed the volume of a sphere of radius R.

Problem 6: Find the volume of the solid cylinder E whose height is 4 and whose base is the disk $\{(r, \theta) : 0 \le r \le 2\cos(\theta)\}$.



Solution: We first look at the cross section of E in the xy−plane to help us determine our bounds.



$$\text{Volume}(E) = \iiint_E 1dV = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} rdrd\theta dz \tag{59}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^{2\cos(\theta)} d\theta dz = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta)d\theta dz \tag{60}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2\theta)+1)d\theta dz = \int_0^4 (\frac{1}{2}\sin(2\theta)+\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \tag{61}$$

$$= \int_0^4 \pi dz = \boxed{4\pi}. \tag{62}$$

Problem 7: Use triple integration in Cartesian coordinates to find the volume of the tetrahedron S that has its vertices at $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, where $a,b,c>0$.
Hint: One of the faces of the tetrahedron lies on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution: We see that an alternative description of S is that it is the solid bound between the planes $x=0$, $y=0$, $z=0$, and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

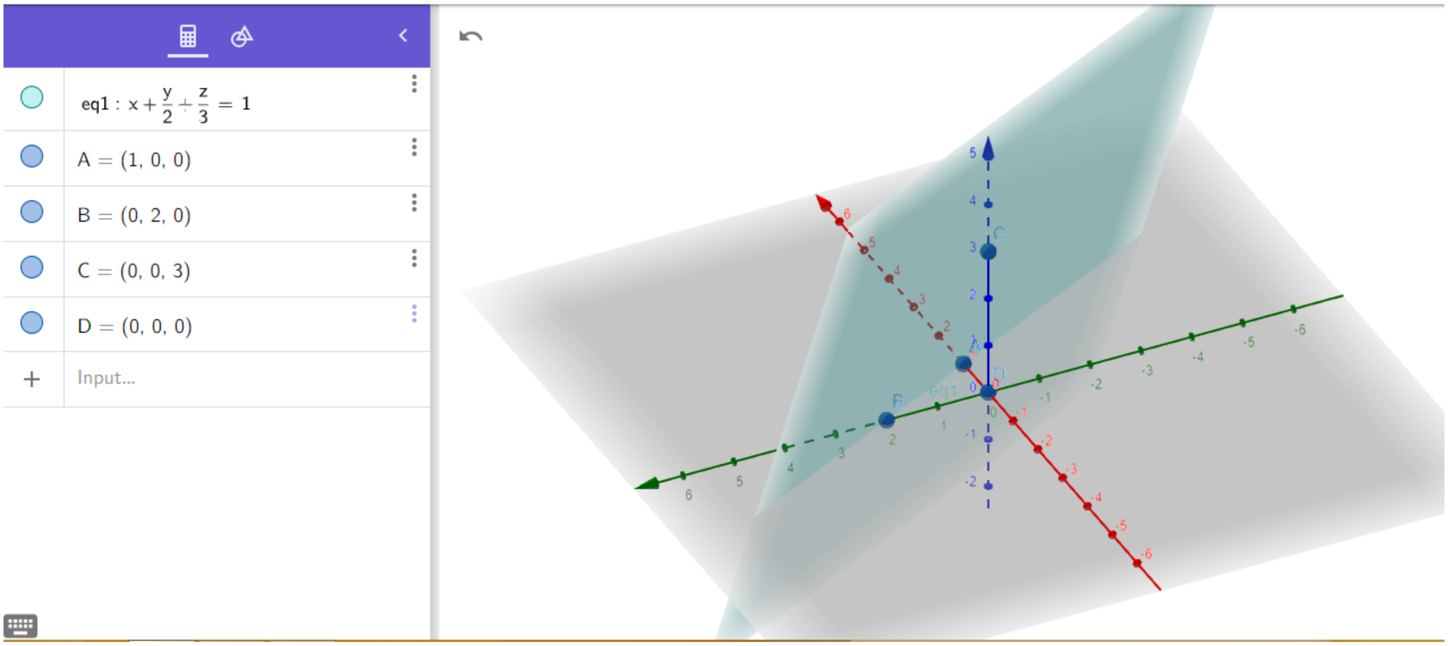


Figure 6: A picture of the solid S when $a=1$, $b=2$, and $c=3$.

$$\text{Volume of } S=\iiint_S 1dV=\int_0^c\int_0^{b(1-\frac{z}{c})}\int_0^{a(1-\frac{z}{c}-\frac{y}{b})}1dxdydz \tag{63}$$

$$=\int_0^c\int_0^{b(1-\frac{z}{c})}a(1-\frac{z}{c}-\frac{y}{b})dydz=a\int_0^c(y-\frac{z}{c}y-\frac{1}{2b}y^2\Big|_{y=0}^{b(1-\frac{z}{c})})dz \tag{64}$$

$$=a\int_0^c\Big(\underbrace{b(1-\frac{z}{c})-\frac{z}{c}b(1-\frac{z}{c})}_{b(1-\frac{z}{c})^2}-\frac{1}{2b}b^2(1-\frac{z}{c})^2\Big)dz=\frac{ab}{2}\int_0^c(1-\frac{z}{c})^2dz \tag{65}$$

$$=\frac{ab}{2}(-\frac{c}{3}(1-\frac{z}{c})^3\Big|_{z=0}^{c})=\boxed{\frac{abc}{6}}. \tag{66}$$