Problem 1: Suppose that the second partial derivative of f are continuous on $R = \{(x,y) : 0 \le x \le a, 0 \le y \le b\}$. Show that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = f(a,b) - f(a,0) - f(0,b) + f(0,0). \tag{1}$$

Hint: Think about the fundamental theorem of calculus.

Solution: We see that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = \int_0^b \int_0^a \frac{\partial^2 f}{\partial x \partial y}(x,y)dxdy = \int_0^b \frac{\partial f}{\partial y}(x,y)\Big|_{x=0}^a dy \tag{2}$$

$$= \int_0^b \left(\frac{\partial f}{\partial y}(a,y) - \frac{\partial f}{\partial y}(0,y)\right) dy = (f(a,y) - f(0,y))\Big|_0^b. \tag{3}$$

$$= f(a,b) - f(a,0) - f(0,b) + f(0,0). \tag{4}$$

Alternatively, since the second partial derivatives of f are continuous on R, we can use Clairaut's Theorem to perform the calculations in the following fashion.

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = \int_0^a \int_0^b \frac{\partial^2 f}{\partial y \partial x}(x,y)dydx = \int_0^a \frac{\partial f}{\partial x}(x,y)\Big|_{y=0}^b dx \tag{5}$$

$$= \int_0^a \left(\frac{\partial f}{\partial x}(x,b) - \frac{\partial f}{\partial x}(x,0)\right) dx = (f(x,b) - f(x,0))\Big|_0^a. \tag{6}$$

$$= f(a,b) - f(0,b) - f(a,0) + f(0,0). \tag{7}$$

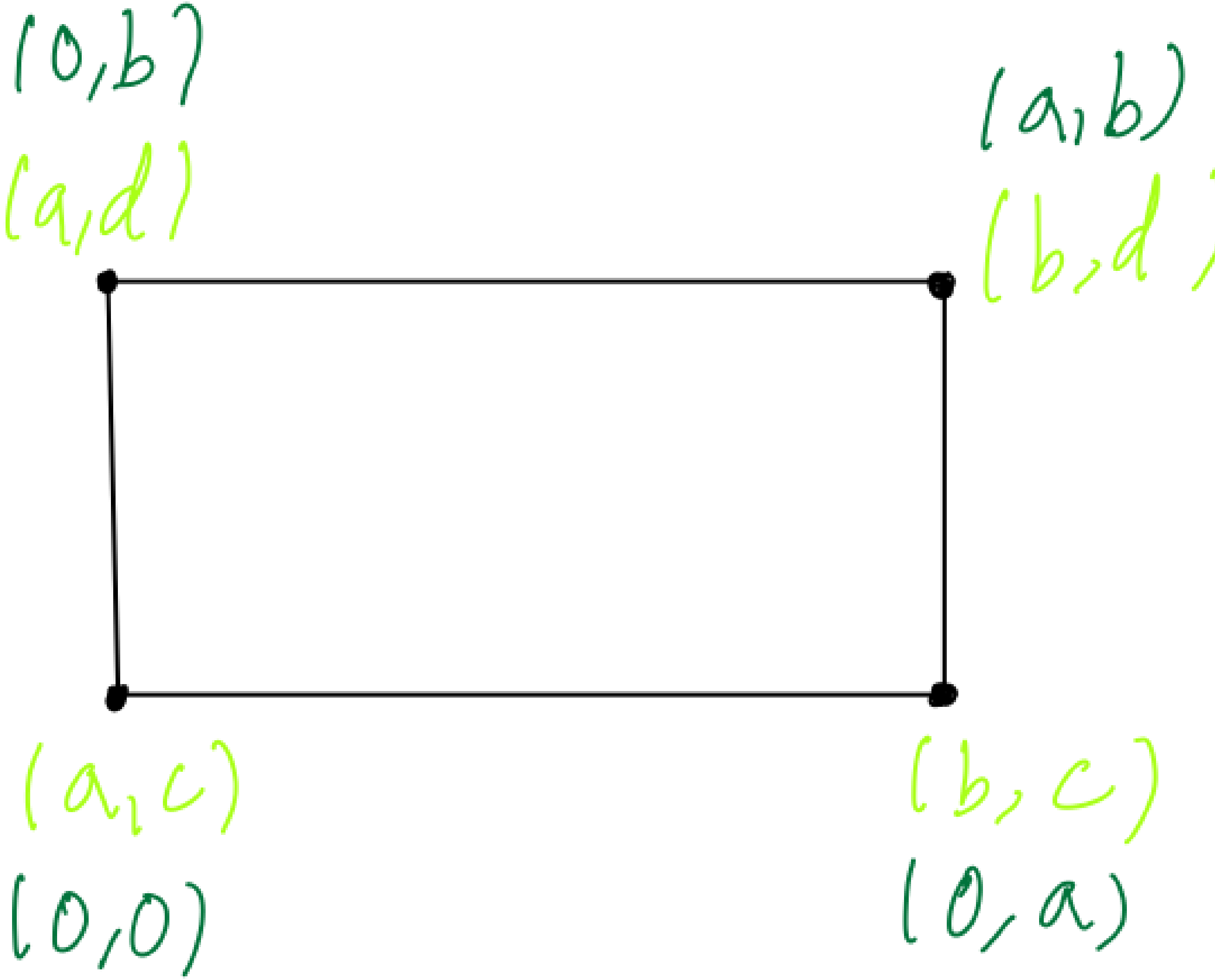
Remark: A similar method can show that if $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, then

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = f(b,d) - f(a,d) - f(b,c) + f(a,c). \tag{8}$$

The Fundamental Theorem of Calculus told us that

$$\int_a^b \frac{df}{dx}(x)dx = f(b) - f(a). \tag{9}$$

Comparing equations (9) and (8), we see that instead taking the difference at the 2 endpoints of a line segment, we are adding 2 opposite corners of the rectangular region R ($f(b,d)$ and $f(a,c)$, or $f(a,b)$ and $f(0,0)$ from the original problem) and subtracting from that the sum of the other 2 opposite corners ($f(a,d)$ and $f(b,c)$, or $f(a,0)$ and $f(0,b)$ from the original problem).


(0,b)
(a,d)
(a,b)
(b,d)
(a,c)
(0,0)
(b,c)
(0,a)

Problem 2: Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

(a) Evaluate $\iint_R \cos(x\sqrt{y})dA$.

(b) Evaluate $\iint_R x^3 y\cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

Solution to a: Noting that $\int \cos(cx)dx$ is easily computable, but $\int \cos(c\sqrt{y})dy$ is not easily computable, we decide to use the order of integration given by $dA = dxdy$. It follows that

$$\iint_R \cos(x\sqrt{y})dA = \int_0^1 \int_0^1 \cos(x\sqrt{y})dxdy \tag{10}$$

$$\overset{u=x\sqrt{y}}{=} \int_0^1 \int_0^1 \frac{\cos(x\sqrt{y})}{\sqrt{y}}\sqrt{y}dxdy \overset{u=x\sqrt{y}}{=} \int_0^1 \int_{x=0}^1 \frac{\cos(u)}{\sqrt{y}}dudy \tag{11}$$

$$= \int_0^1 \left(\frac{\sin(u)}{\sqrt{y}}\Big|_{x=0}^1\right)dy = \int_0^1 \left(\frac{\sin(x\sqrt{y})}{\sqrt{y}}\Big|_{x=0}^1\right)dy = \int_0^1 \frac{\sin(\sqrt{y})}{\sqrt{y}}dy \tag{12}$$

$$\overset{u=\sqrt{y}}{=} \int_0^1 2\sin(\sqrt{y})\frac{dy}{2\sqrt{y}} \overset{u=\sqrt{y}}{=} \int_{y=0}^1 2\sin(u)du = -2\cos(u)\Big|_{y=0}^1 \tag{13}$$

$$= -2\cos(\sqrt{y})\Big|_{y=0}^1 = \boxed{2-2\cos(1)}. \tag{14}$$

Solution to b: Noting that $\int c_1x^3\cos(c_2x^2)dx$ is not easily computable, but $\int c_1y\cos(c_2y^2)dy$ is easily computable, we decide to use the order of integration given by $dA = dydx$. It follows that

$$\iint_R x^3y\cos(x^2y^2)dA = \int_0^1 \int_0^1 x^3y\cos(x^2y^2)dydx \tag{15}$$

$$\overset{u=y^2}{=} \int_0^1 \int_0^1 \frac{x^3}{2}\cos(x^2y^2)2ydydx \overset{u=y^2}{=} \int_0^1 \int_{y=0}^1 \frac{x^3}{2}\cos(x^2u)dudx \tag{16}$$

$$\overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(x^2u)x^2dudx \overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(v)dvdx \tag{17}$$

. . . .

$$= \int_0^1 \left(\frac{x}{2}\sin(v)\Big|_{y=0}^1\right)dx = \int_0^1 \left(\frac{x}{2}\sin(x^2u)\Big|_{y=0}^1\right)dx \tag{18}$$

. . . .

$$= \int_0^1 \left(\frac{x}{2}\sin(x^2y^2)\Big|_{y=0}^1\right)dx = \int_0^1 \frac{x}{2}\sin(x^2)dx \overset{u=x^2}{=} \int_0^1 \frac{1}{4}\sin(x^2)2xdx \tag{19}$$

. . . .

$$\overset{u=x^2}{=} \int_{x=0}^1 \frac{1}{4}\sin(u)du = -\frac{1}{4}\cos(u)\Big|_{x=0}^1 \tag{20}$$

. . . .

$$= -\frac{1}{4}\cos(x^2)\Big|_{x=0}^1 = \boxed{\frac{1}{4} - \frac{1}{4}\cos(1)}. \tag{21}$$

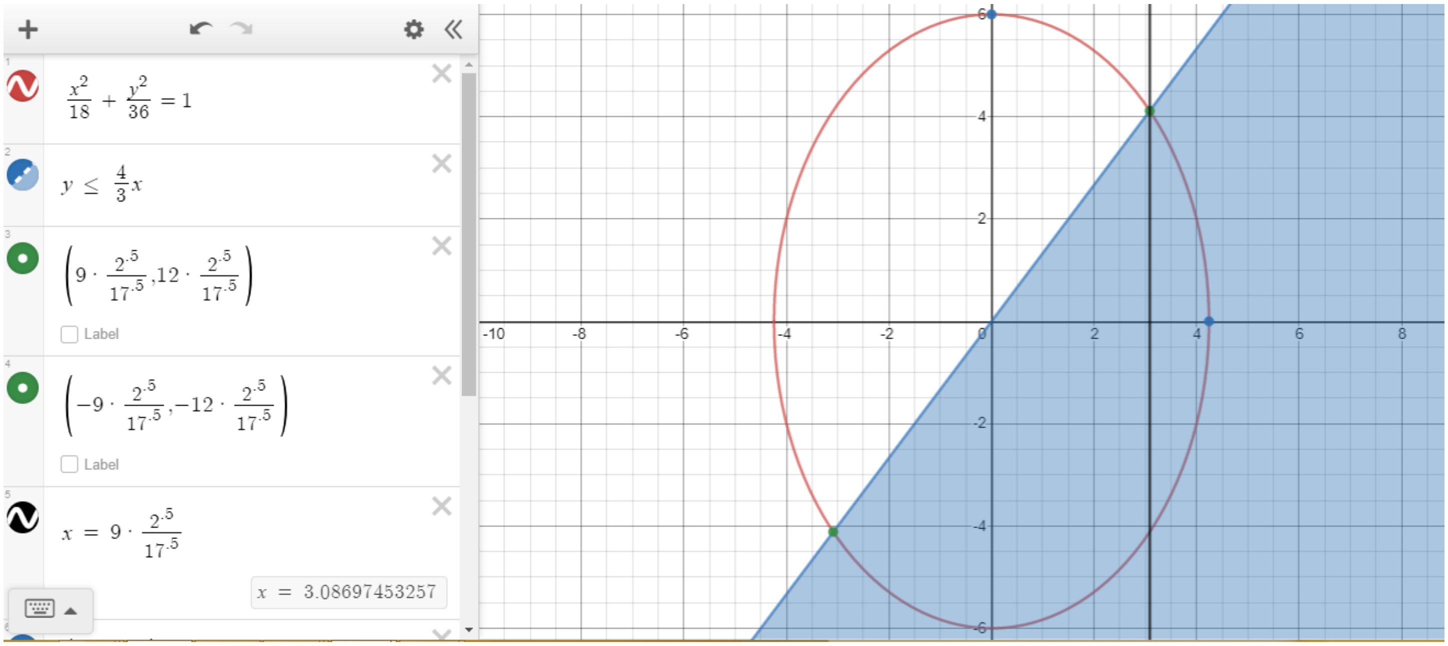
Problem 3: Let R be the region inside of the ellipse $\frac{x^2}{18}+\frac{y^2}{36}=1$ for which we also have $y\le\frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy\,dA. \tag{22}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18}+\frac{y^2}{36}=1$ and $y=\frac{4}{3}x$. We see that

$$\begin{array}{rcl} \frac{x^2}{18}+\frac{y^2}{36} & = & 1 \\ y & = & \frac{4}{3}x \end{array} \to \frac{x^2}{18}+\frac{\frac{16}{9}x^2}{36}=1 \tag{23}$$

$$\to x=\pm\frac{9\sqrt{2}}{\sqrt{17}}\to(x,y)=(-\frac{9\sqrt{2}}{\sqrt{17}},-\frac{12\sqrt{2}}{\sqrt{17}}),(\frac{9\sqrt{2}}{\sqrt{17}},\frac{12\sqrt{2}}{\sqrt{17}}). \tag{24}$$

We now see that the area of R is

$$\iint_R 1dA=\iint_R 1dydx \tag{25}$$

....

$$=\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x}1dydx+\int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}}1dydx \tag{26}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{27}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx \tag{28}$$

Since

$$\int \sqrt{1-x^2} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + C, \quad \text{(substitute } x = \sin(\theta)) \tag{29}$$

we see that

$$\int \sqrt{36-2x^2}dx = \int 6\sqrt{1-(\frac{x}{3\sqrt{2}})^2}dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1-y^2}dy \tag{30}$$

. . . .

$$= 9\sqrt{2}y\sqrt{1-y^2} + 9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}). \tag{31}$$

Applying this result to equation (28), we see that

$$\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx \tag{32}$$

. . . .

$$= \left(\frac{2}{3}x^2 + \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} + \left(x\sqrt{36-2x^2} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{33}$$

. . . .

$$2\left(\frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{34}$$

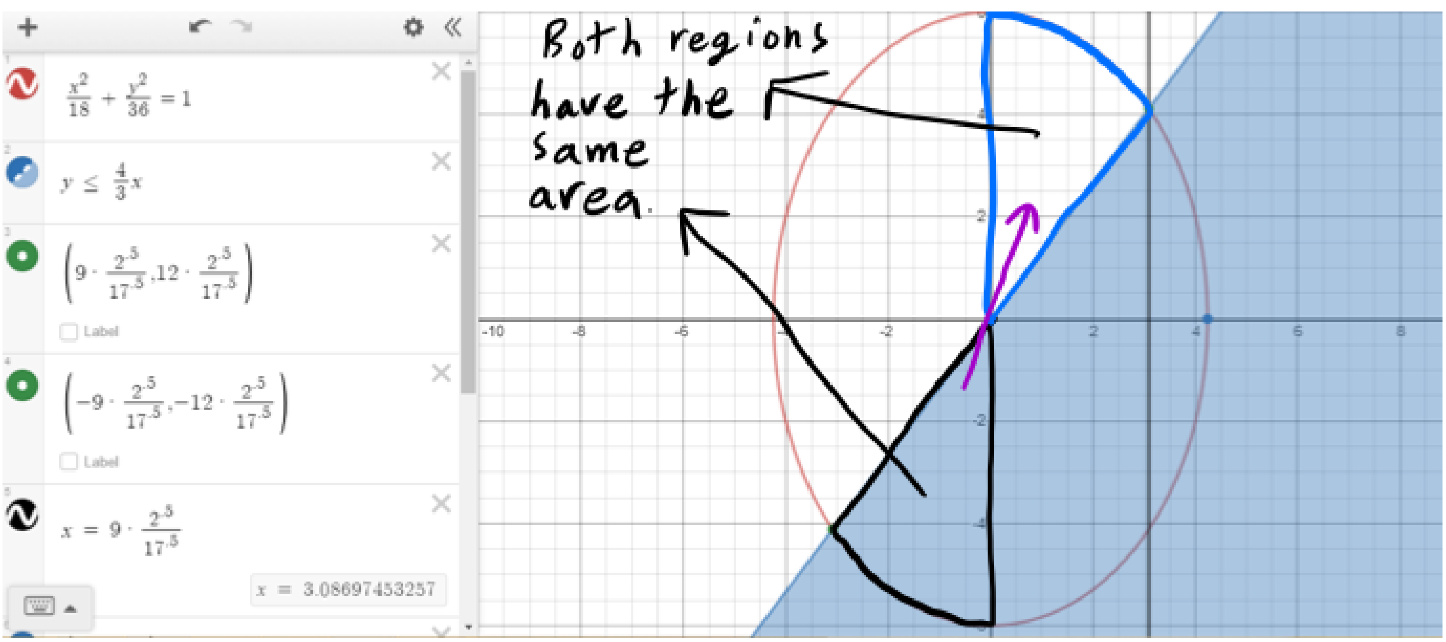
. . . .

$$x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+x\sqrt{36-2x^2}\Big|_{3\sqrt{2}}$$
$$-x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}}-18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \quad (35)$$

$$=x\sqrt{36-2x^2}\Big|_{3\sqrt{2}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}} \quad (36)$$

$$=0+18\sqrt{2}\sin^{-1}(1)=\boxed{9\sqrt{2}\pi}. \quad (37)$$

Remark: For the ellipse $\frac{y^2}{36}+\frac{x^2}{18}=1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6\cdot 3\sqrt{2}\cdot\pi=18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x,y) across the origin we get the point $(-x,-y)$, and that reflection across the origin (or reflection across any other point) preserves area as shown in the picture below.



Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy\,dA=\iint_R xy\,dy\,dx \quad (38)$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xydydx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xydydx \tag{39}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{40}$$

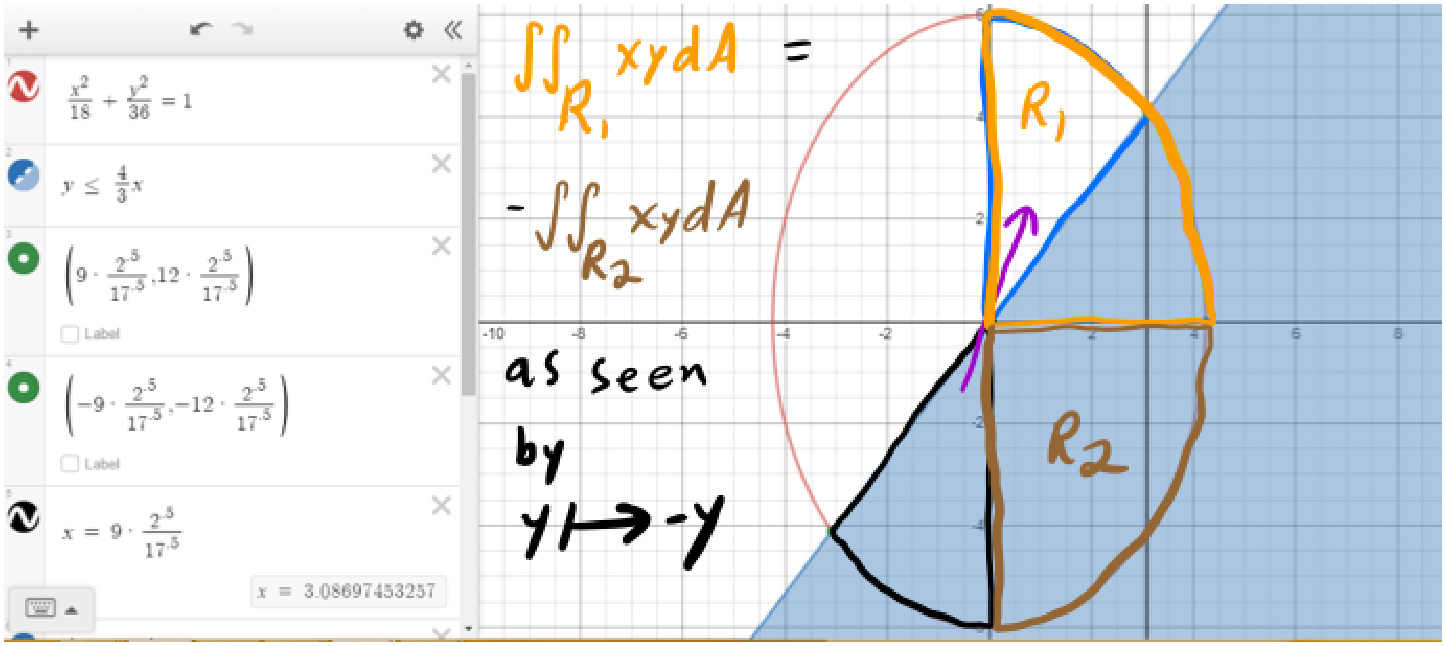
. . . .

$$\begin{aligned} = &\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \\ &+ \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \end{aligned} \tag{41}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{42}$$

Remark: We see that both integrals appearing in equation (39) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0 as shown in the picture below.


$\frac{x^2}{18} + \frac{y^2}{36} = 1$
$y \leq \frac{4}{3}x$
$\left(9 \cdot \frac{2^{.5}}{17^{.5}}, 12 \cdot \frac{2^{.5}}{17^{.5}}\right)$
Label
$\left(-9 \cdot \frac{2^{.5}}{17^{.5}}, -12 \cdot \frac{2^{.5}}{17^{.5}}\right)$
Label
$x = 9 \cdot \frac{2^{.5}}{17^{.5}}$
x = 3.08697453257
$\iint_{R_1} xy\,dA =$
$-\iint_{R_2} xy\,dA$
as seen
by
$y \mapsto -y$
R_1
R_2

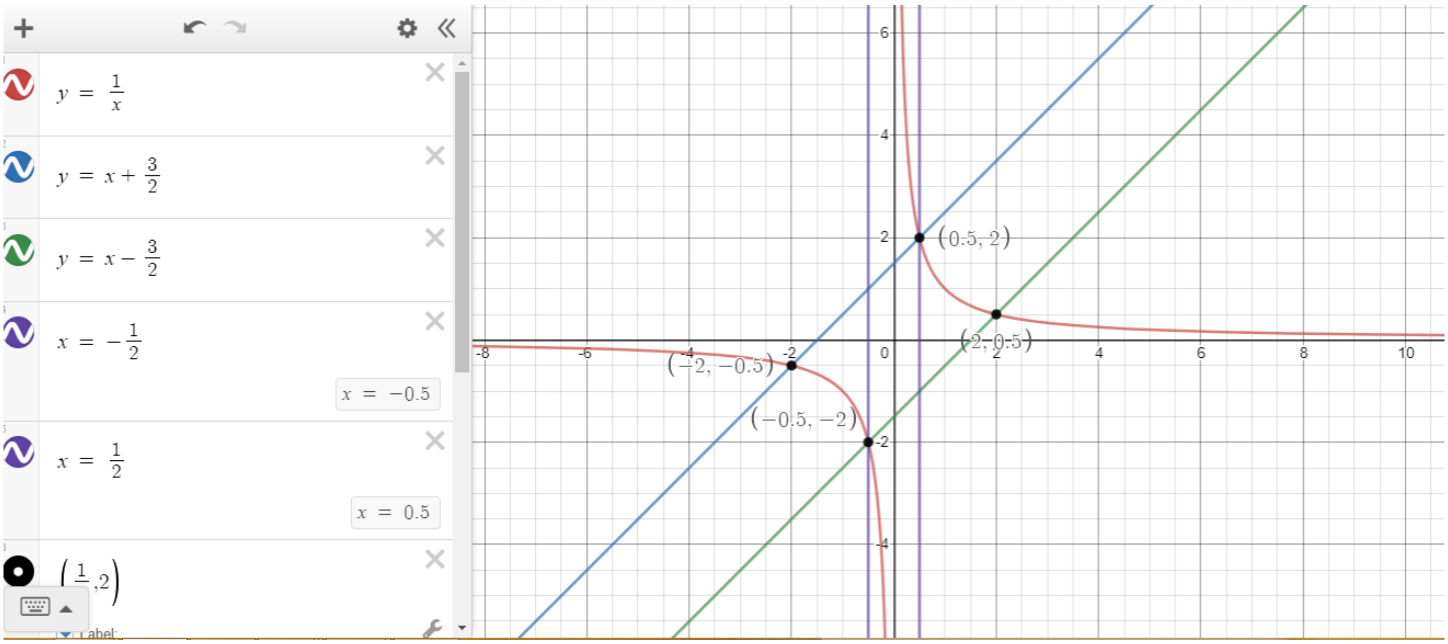
Problem 4: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy dA. \tag{43}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

$$\begin{matrix} y & = & \frac{1}{x} \\ y & = & x + \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x + \frac{3}{2} \rightarrow x^2 + \frac{3}{2}x - 1 = 0 \tag{44}$$

$$\rightarrow x = -2, \frac{1}{2} \rightarrow (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2). \tag{45}$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

$$\begin{matrix} y & = & \frac{1}{x} \\ y & = & x - \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x - \frac{3}{2} \rightarrow x^2 - \frac{3}{2}x - 1 = 0 \tag{46}$$

$$\rightarrow x = -\frac{1}{2}, 2 \rightarrow (x, y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}). \tag{47}$$

We now see that the area of R is

$$\iint_R 1dA = \iint_R 1dydx \tag{48}$$

$$= \int_{-2}^{-\frac{1}{2}}\int_{\frac{1}{x}}^{x+\frac{3}{2}} 1dydx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1dydx + \int_{\frac{1}{2}}^{2}\int_{x-\frac{3}{2}}^{\frac{1}{x}} 1dydx \tag{49}$$

$$= \int_{-2}^{-\frac{1}{2}}\left(y\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}}\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(y\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}}\right)dx + \int_{\frac{1}{2}}^{2}\left(y\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}}\right)dx \tag{50}$$

$$= \int_{-2}^{-\frac{1}{2}}\left(x+\frac{3}{2}-\frac{1}{x}\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3dx + \int_{\frac{1}{2}}^{2}\left(\frac{1}{x}-x+\frac{3}{2}\right)dx \tag{51}$$

$$\left(\frac{1}{2}x^2+\frac{3}{2}x-\ln|x|\right)\Big|_{-2}^{-\frac{1}{2}} + 3x\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \left(\ln|x|-\frac{1}{2}x^2+\frac{3}{2}x\right)\Big|_{\frac{1}{2}}^{2} \tag{52}$$

$$= (1+2\ln(2)-\frac{5}{8})+3+(1+2\ln(2)-\frac{5}{8}) = \boxed{\frac{15}{4}+4\ln(2)}. \tag{53}$$

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xydA = \iint_R xydydx \tag{54}$$

$$= \int_{-2}^{-\frac{1}{2}}\int_{\frac{1}{x}}^{x+\frac{3}{2}} xydydx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xydydx + \int_{\frac{1}{2}}^{2}\int_{x-\frac{3}{2}}^{\frac{1}{x}} xydydx \tag{55}$$

$$= \int_{-2}^{-\frac{1}{2}}\left(\frac{1}{2}xy^2\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}}\right)dx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}}\right)dx + \int_{\frac{1}{2}}^{2}\left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}}\right)dx \tag{56}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(\frac{1}{x})^2\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}x(\frac{1}{x})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx \quad (57)$$

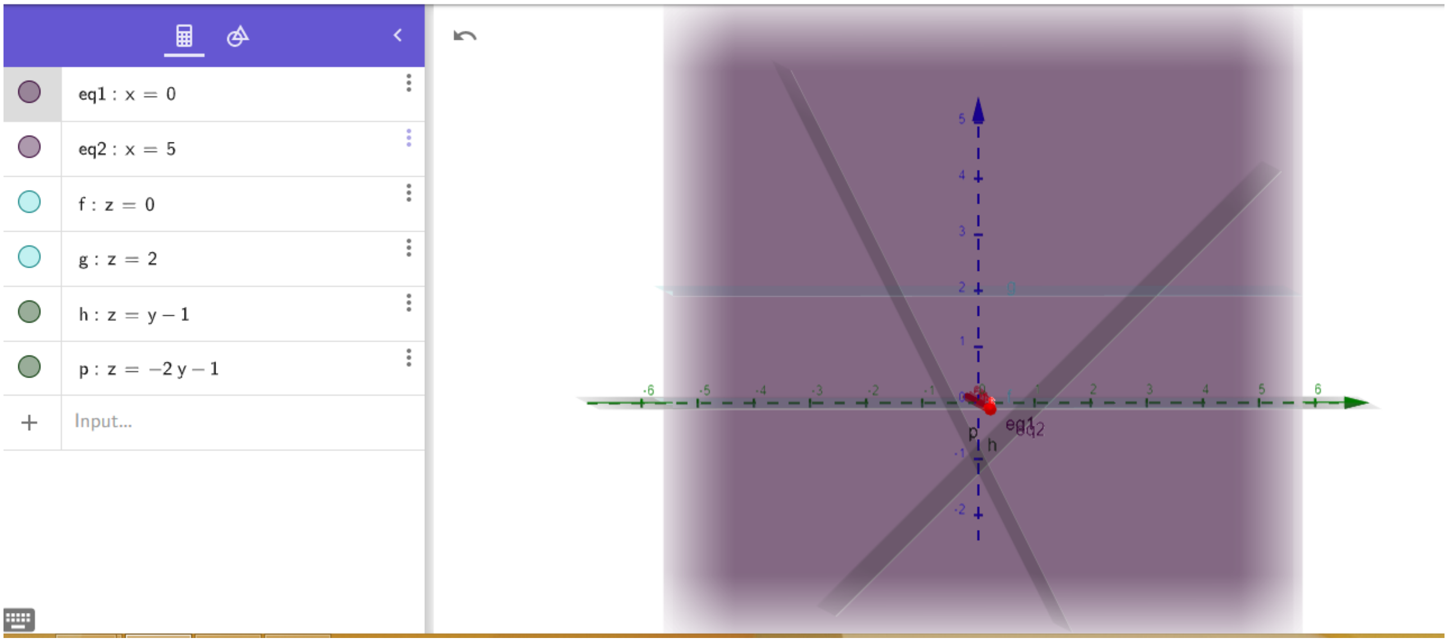
$$= \frac{1}{2}\int_{-2}^{-\frac{1}{2}} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x}\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2 dx + \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x^3 + 3x^2 - \frac{9}{4}x\right) dx \quad (58)$$

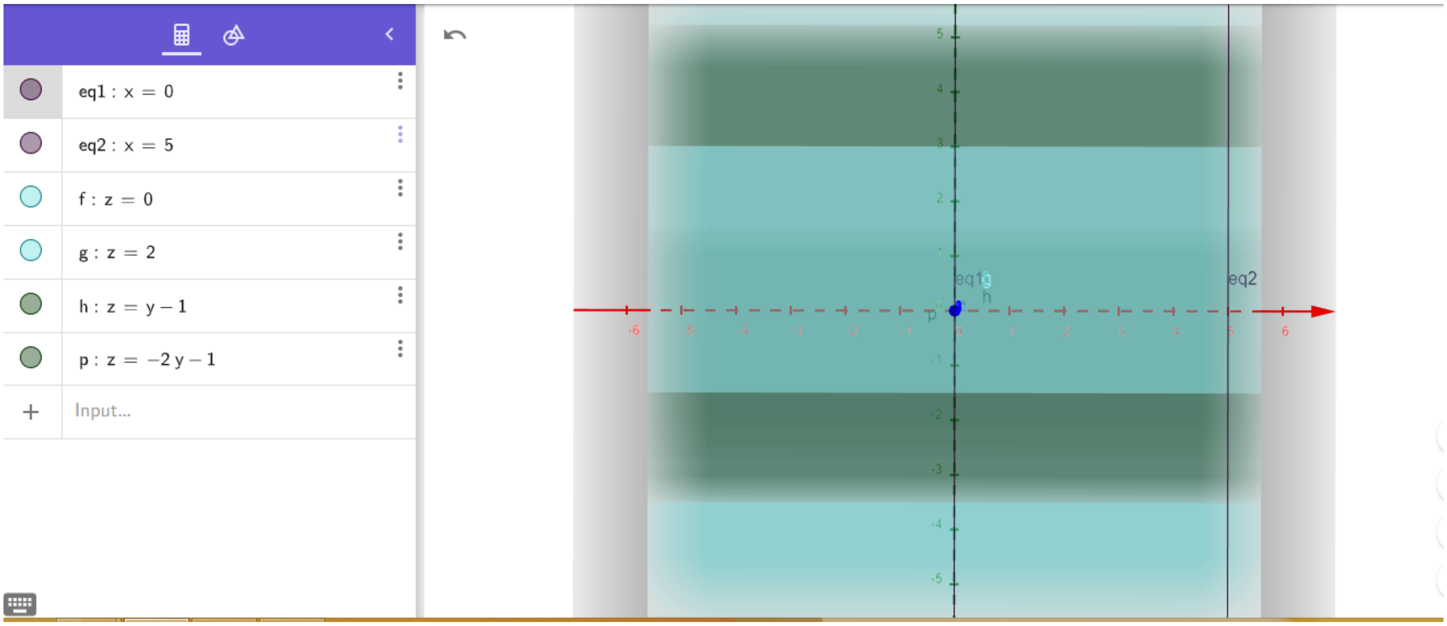
$$= \frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x|\right)\Big|_{-2}^{-\frac{1}{2}} + x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2}\left(\ln|x| - \frac{1}{4}x^4 + x^3 - \frac{9}{8}x^2\right)\Big|_{\frac{1}{2}}^{2} \quad (59)$$

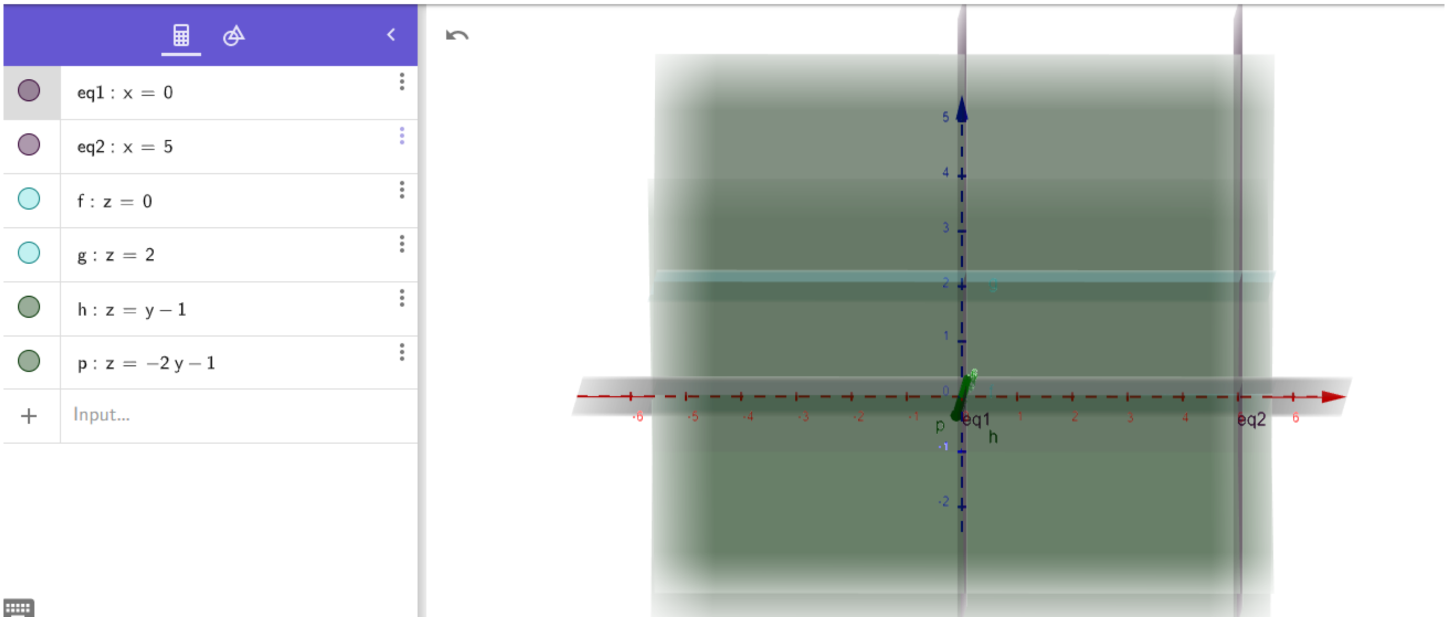
$$= \boxed{2\ln(2) - \frac{5}{64}} \quad (60)$$

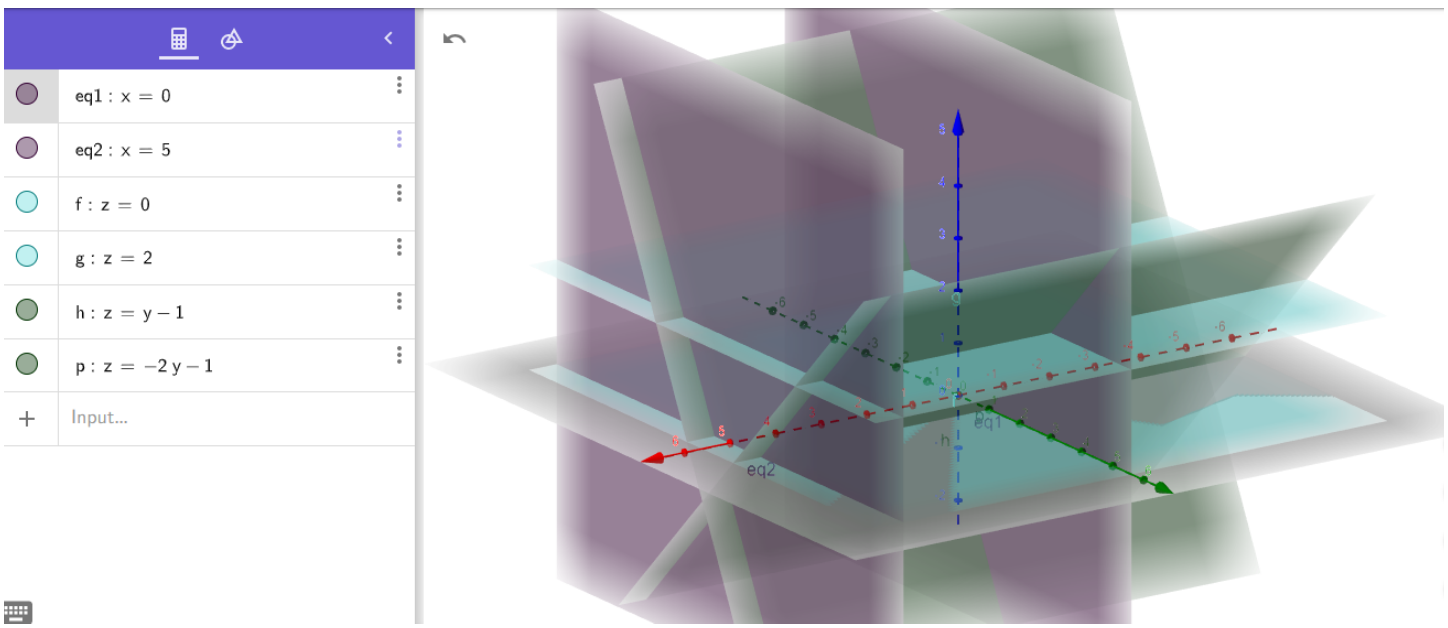
Problem 5: Find the volume of the solid bounded by the planes $x = 0, x = 5, z = y - 1, z = -2y - 1, z = 0$, and $z = 2$.

Solution: Let us first examine our solid from a few different angles.







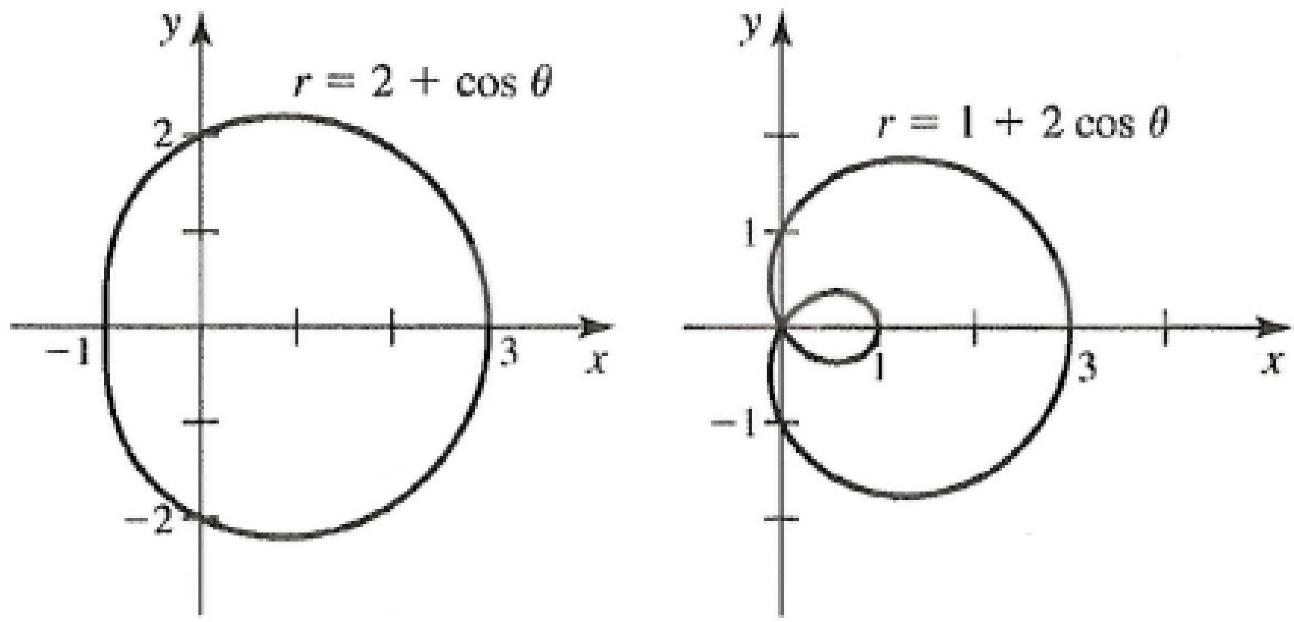


Due to the third and fourth pictures, we will choose to view the 'base' of our solid in the xz-plane so that it is simply the rectangle $R = \{(x, z) \in \mathbb{R}^2 \mid 0 \le x \le 5,\ 0 \le z \le 2\}$. We could also come to this decision simply by examining the x and z bounds without drawing any diagrams. We then see that the 'heights' of our solid are along the y-axis. Solving for y in terms of x and z we see that $y = z + 1$ and $y = -\frac{z+1}{2}$ are the surfaces bounding the 'heights' of our solid. By examining the values of y for some $(x, z) \in R$ (such as $(0, 0)$), we see that $y = z + 1$ is the upper bound for our heights and $y = \frac{z+1}{2}$ is the lower bound for our heights. We now see that the volume V of our solid is given by

$$V = \iint_R (y_{\text{top}} - y_{\text{bottom}})\, dA = \iint_R \left(z + 1 - (-\frac{z+1}{2})\right) dA \qquad (61)$$

$$= \int_0^5 \int_0^2 3\frac{z+1}{2}dzdx = \int_0^5 \left(\frac{3}{4}z^2 + \frac{3}{2}z\right)\Big|_{z=0}^2 dx = \int_0^5 6dx = \boxed{30}. \tag{62}$$

Problem 6: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.



(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.

(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.

(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Solution to (a): Letting R denote the interior of the limaçon $r = 2 + \cos(\theta)$, we see that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} rdrd\theta \tag{63}$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi} \frac{1}{2}(2+\cos(\theta))^2 d\theta \tag{64}$$

$$= \int_0^{2\pi}(2 + 2\cos(\theta) + \frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi}(2 + 2\cos(\theta) + \frac{1}{4}\cos(2\theta) + \frac{1}{4})d\theta \tag{65}$$

$$(\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}. \tag{66}$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\int_0^{1+2\cos(\theta)} rdrd\theta \tag{67}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos(\theta))^2 d\theta \tag{68}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + 2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + \cos(2\theta) + 1)d\theta \tag{69}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}. \tag{70}$$

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi}\int_0^{1+2\cos(\theta)} rdrd\theta \tag{71}$$

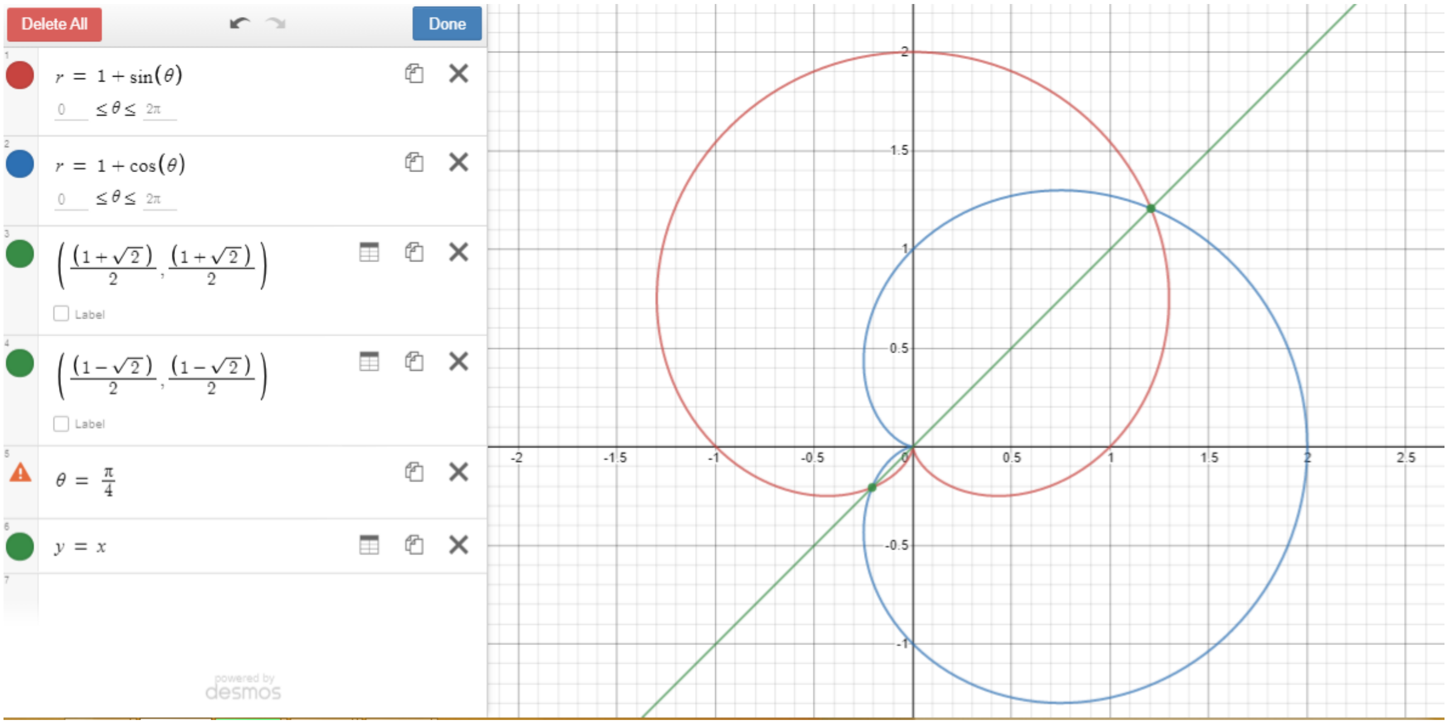
$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{72}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{73}$$

Problem 7: Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1 + \cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.

Solution: We begin by drawing a picture of the region R.



We see that the 2 cardiods intersect when $\sin(\theta) = \cos(\theta)$, which occurs when $\theta = \frac{\pi}{4}, -\frac{3\pi}{4}$. We now see that when $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$ we have $1 + \sin(\theta) \le 1 + \cos(\theta)$ and when $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$ we have $1 + \cos(\theta) \le 1 + \sin(\theta)$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_0^{1+\sin(\theta)} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos(\theta)} rdrd\theta \tag{74}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\sin(\theta)} d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\cos(\theta)} d\theta \tag{75}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \sin^2(\theta))d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \cos^2(\theta))d\theta \tag{76}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \frac{1 - \cos(2\theta)}{2})d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \frac{1 + \cos(2\theta)}{2})d\theta \tag{77}$$

$$= \left(\frac{3}{4}\theta - \cos(\theta) + \frac{-\sin(2\theta)}{4}\Bigg|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\right) + \left(\frac{3}{4}\theta + \sin(\theta) + \frac{\sin(2\theta)}{4}\Bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right) \tag{78}$$

. . . .

$$= \boxed{\frac{3\pi}{2} - 2\sqrt{2}}. \tag{79}$$

Problem 8: Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

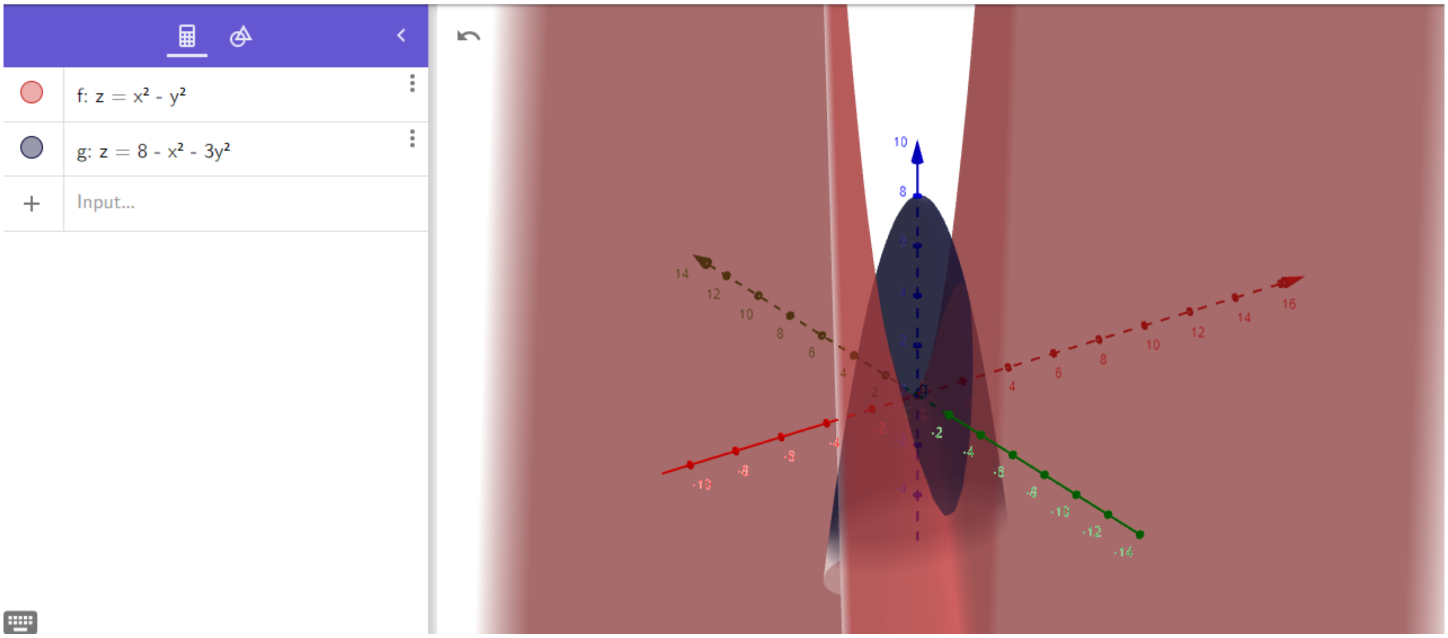


Figure 1: A view of the solid S whose volume we are calculating.

Solution: We begin by finding the (x, y)-coordinates of the curves of intersection of the 2 given surfaces. We see that

$$8 - x^2 - 3y^2 = z = x^2 - y^2 \to 2x^2 + 2y^2 = 8 \to x^2 + y^2 = 4, \tag{80}$$

so the (x, y)-coordinates of the curve of intersection is simply the circle of radius 2 centered at the origin.

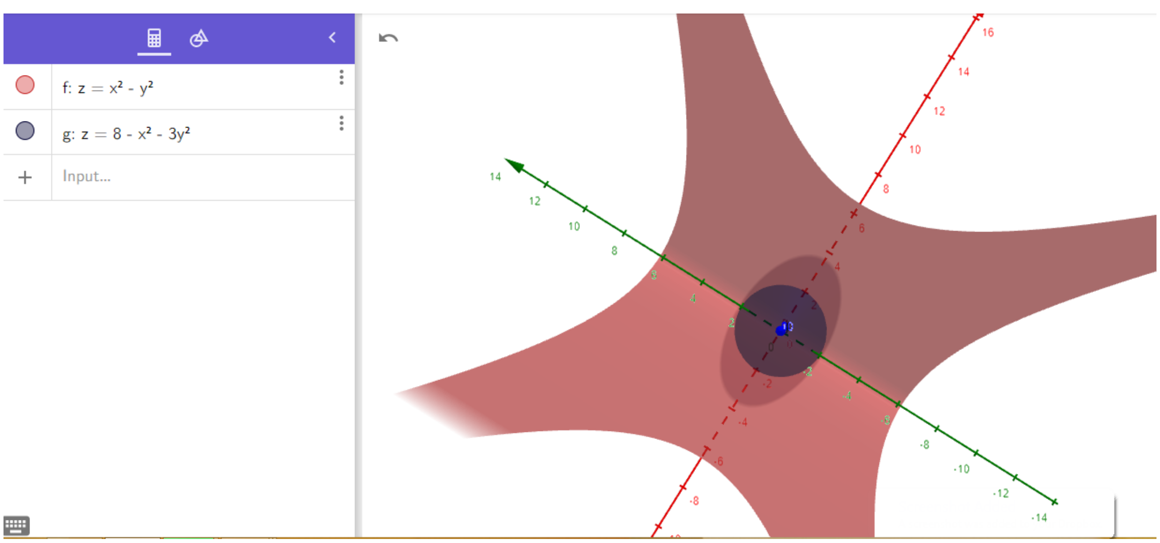


Figure 2: A bird's eye view of the solid S that is used to find the region of integration R.

Noting that $8 - 0^2 - 3 \cdot 0^2 = 8 > 0 = 0^2 - 0^2$, we see that the curve $z = 8 - x^2 - 3y^2$ lies above the curve $z = x^2 - y^2$ for all (x, y) inside of R, the disc of radius 2 centered at the origin. We now see that

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R \Big((8 - x^2 - 3y^2) - (x^2 - y^2)\Big)dA \tag{81}$$

. . . .

$$= \iint_R (8 - 2x^2 - 2y^2)dA = \int_0^{2\pi} \int_0^2 (8 - 2r^2) r dr d\theta \tag{82}$$

. . . .

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r - 2r^3)dr\right) = (2\pi)\left(4r^2 - \frac{1}{2}r^4\Big|_0^2\right) = \boxed{16\pi}. \tag{83}$$