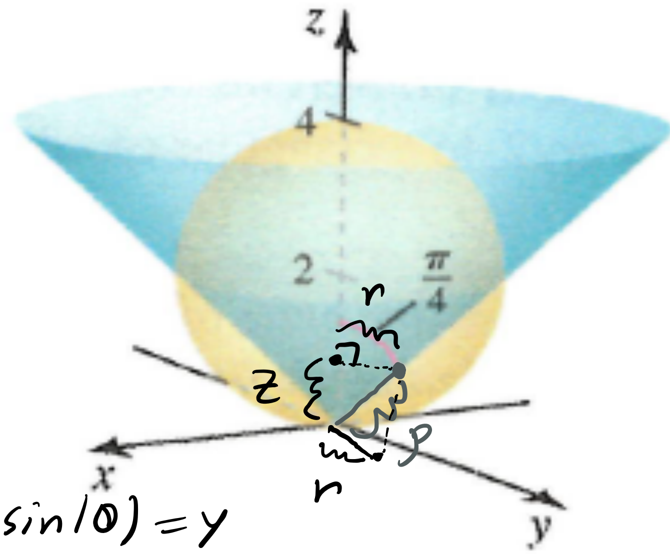
Problem 3: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.

Similar to $z = \sqrt{x^2+y^2}$

$$\rho^2 = x^2+y^2+z^2 = r^2+z^2$$



$$\rho\cos(\varphi) = z$$

$$\rho\sin(\varphi) = r$$

$$\rho\sin(\varphi)\sin(\theta) = r\sin(\theta) = y$$

$$\rho\sin(\varphi)\cos(\theta) = r\cos(\theta) = x$$

$$\rho = 4\cos(\varphi)$$

$$\rho^2 = 4\rho\cos(\varphi) \to x^2+y^2+z^2 = 4z$$

$$\to x^2+y^2+z^2-4z+4 = 4$$

$$\to x^2+y^2+(z-2)^2 = 4 \to$$ the sphere of radius 2 centered at $(0,0,2)$

$$\mathrm{Vol}(S) = \iiint_S 1\,dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\varphi\right)$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{4\cos(\varphi)} \rho^2 \sin(\varphi)\, d\rho\, d\varphi$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\pi/2} \sin(\varphi) \left(\int_{0}^{4\cos(\varphi)} \rho^2\, d\rho \right) d\varphi$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\pi/2} \sin(\varphi) \left(\frac{\rho^3}{3} \Big|_{\rho=0}^{4\cos(\varphi)} \right) d\varphi$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \sin(\varphi)\, \frac{64\cos^3(\varphi)}{3}\, d\varphi$$

$u = \cos(\varphi)$

$du = -\sin(\varphi)\, d\varphi$

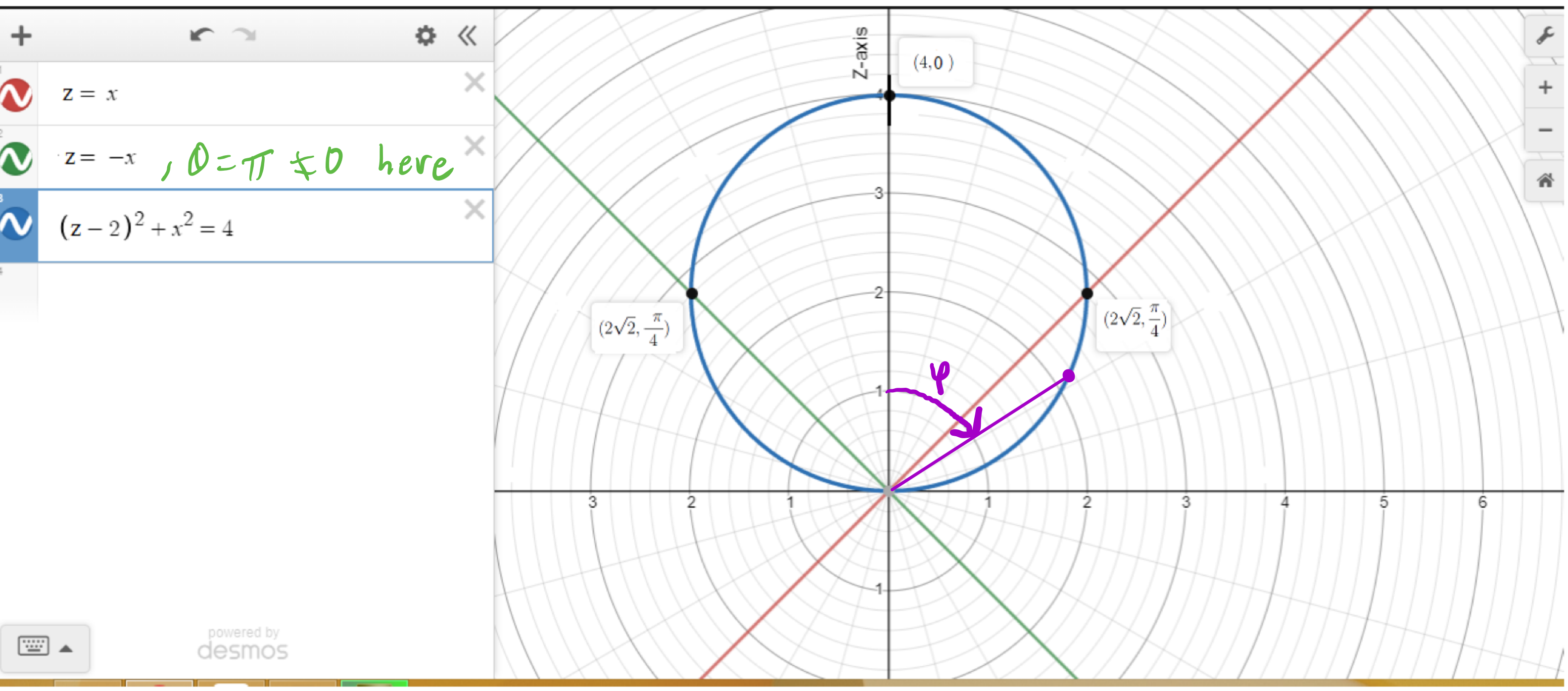


Figure 2: The xz-plane cross section in spherical coordinates.

$$= 2\pi \int_{\varphi=\frac{\pi}{4}}^{\pi/2} \frac{64}{3} u^3 (-du)$$

$$= -\frac{128\pi}{3} \left(\frac{u^4}{4} \Bigg|_{\varphi=\frac{\pi}{4}}^{\pi/2} \right)$$

$$= -\frac{128\pi}{3} \left(\frac{\cos^4(\varphi)}{4} \Bigg|_{\frac{\pi}{4}}^{\pi/2} \right)$$

$$= \frac{-128\pi}{3} \left(0 - \frac{\left(\frac{1}{\sqrt{2}}\right)^4}{4} \right)$$

$$= -\frac{128\pi}{3} \left(-\frac{1}{16} \right) = \boxed{\frac{8\pi}{3}}$$

Problem 3: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.

$\rho^2 = x^2+y^2+z^2$
$= r^2+z^2$

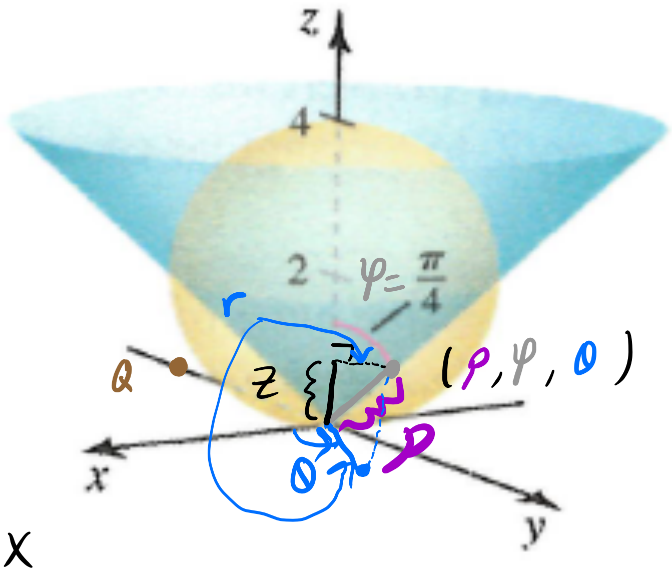
$0 \leq \varphi \leq \pi$

$z = \rho\cos(\varphi)$

$\rho\sin(\varphi) = r$

$\rho\sin(\varphi)\sin(\theta) = r\sin(\theta) = y$

$\rho\sin(\varphi)\cos(\theta) = r\cos(\theta) = x$



$Q = \cancel{\left(2, \frac{3\pi}{2}, \frac{\pi}{2}\right)}$

$= \left(2, \frac{\pi}{2}, \frac{3\pi}{2}\right)$ ✓

$\rho = 4\cos(\varphi) \to \rho^2 = 4\rho\cos(\varphi) \to$

$x^2+y^2+z^2 = 4z \to x^2+y^2+z^2-4z+4 = 4 \to$

$x^2+y^2+(z-2)^2 = 2^2 \to$ sphere of radius 2 centered at $(0,0,2)$.

$r^2+(z-2)^2 = 2^2$

$$\mathrm{Vol}(S) = \iiint_S 1\,dV = \int_0^{2\pi}\int_0^{2}\int_z^{\sqrt{4z-z^2}} r\,dr\,dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}\left(\frac{r^2}{2}\Big|_{r=z}^{\sqrt{4z-z^2}}\right)dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{1}{2}\underbrace{\left(4z - z^2 - z^2\right)}_{2z - z^2} \, dz \, d\theta = \int_0^{2\pi} \left(z^2 - \frac{z^3}{3} \Big|_{z=0}^{2} \right) d\theta$$

$$= \int_0^{2\pi} \frac{4}{3} \, d\theta = \frac{4}{3} \cdot 2\pi = \boxed{\frac{8\pi}{3}}$$

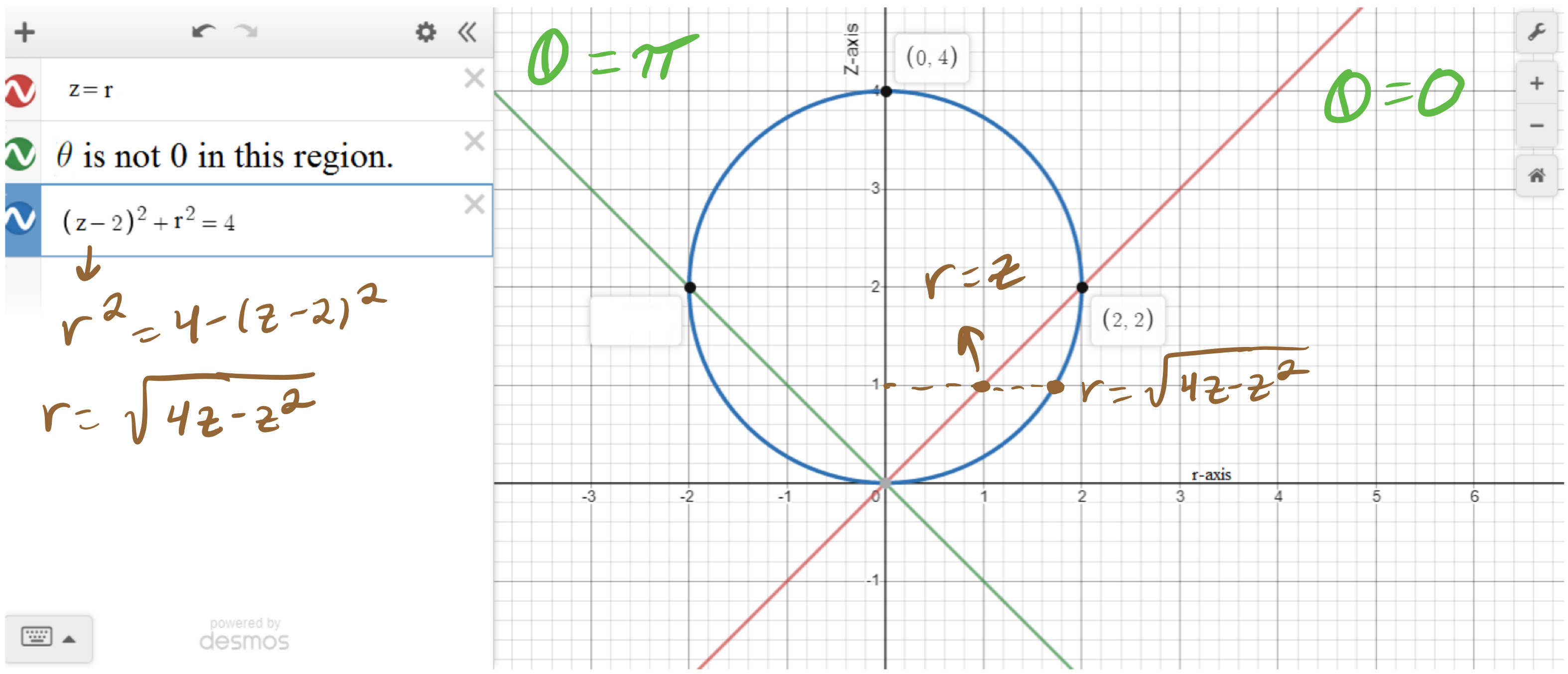


Figure 3: The xz-plane cross section in cylindrical coordinates.

Problem 4: Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.

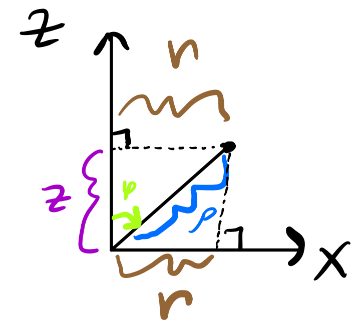
Spherical $\to (\rho, \varphi, \theta)$

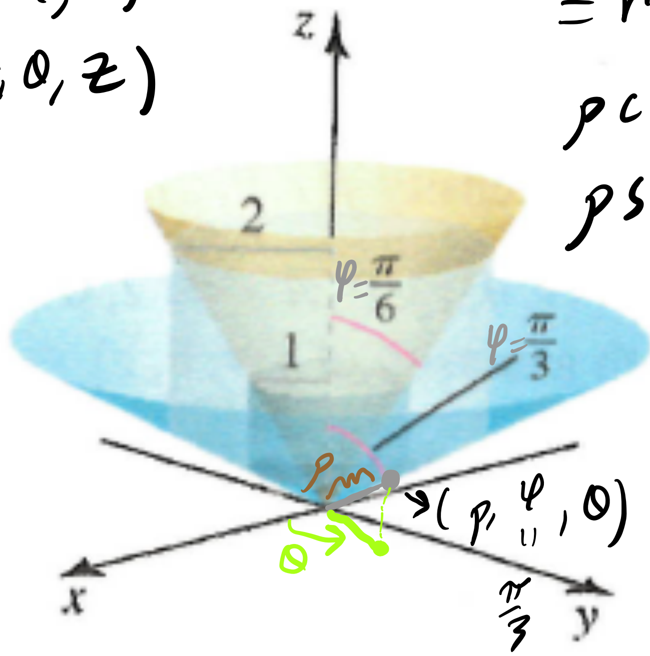
cylindrical $\to (r, \theta, z)$

$$\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2$$

$$\rho\cos(\varphi) = z$$

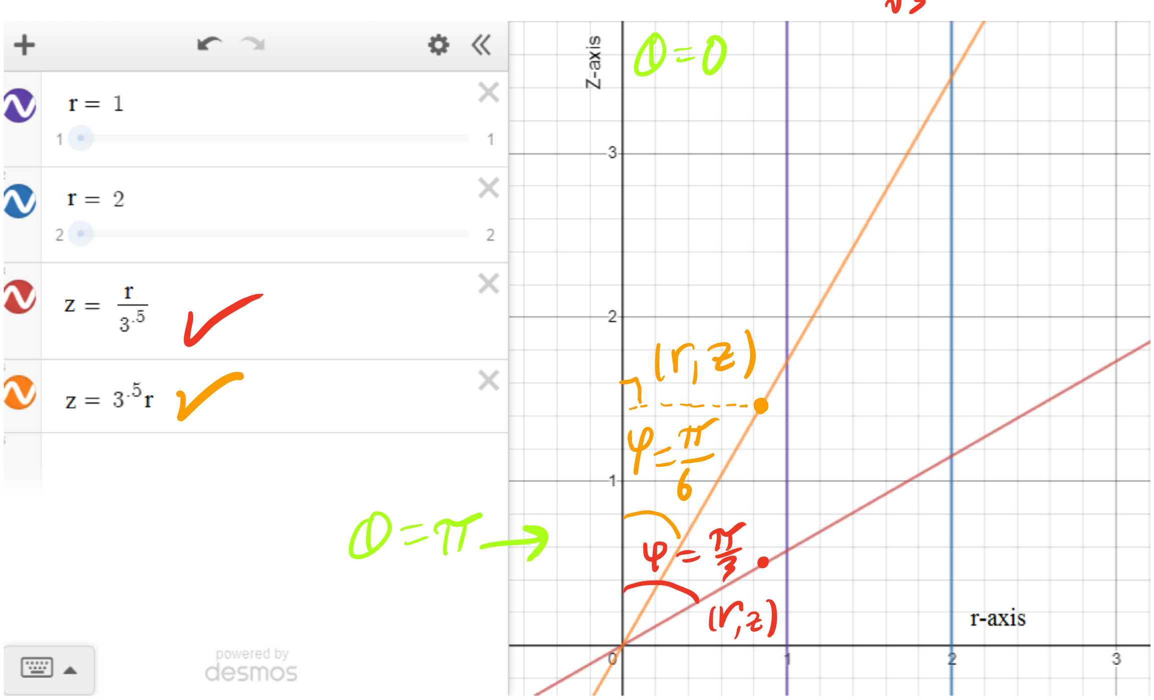
$$\rho\sin(\varphi) = r$$





$$\rho\sin(\varphi)\sin(\theta) = r\sin(\theta) = y, \quad \rho\sin(\varphi)\cos(\theta) = r\cos(\theta) = x$$

$$\mathrm{Vol}(S) = \iiint_S 1\,dV = \int_0^{2\pi}\int_1^2\int_{\frac{r}{\sqrt{3}}}^{r\sqrt{3}} r\,dz\,dr\,d\theta$$



$$\frac{r}{z} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\to z = r\sqrt{3}$$

$$\frac{r}{z} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\to z = \frac{r}{\sqrt{3}}$$

Figure 5: The xz-plane cross section in cylindrical coordinates.

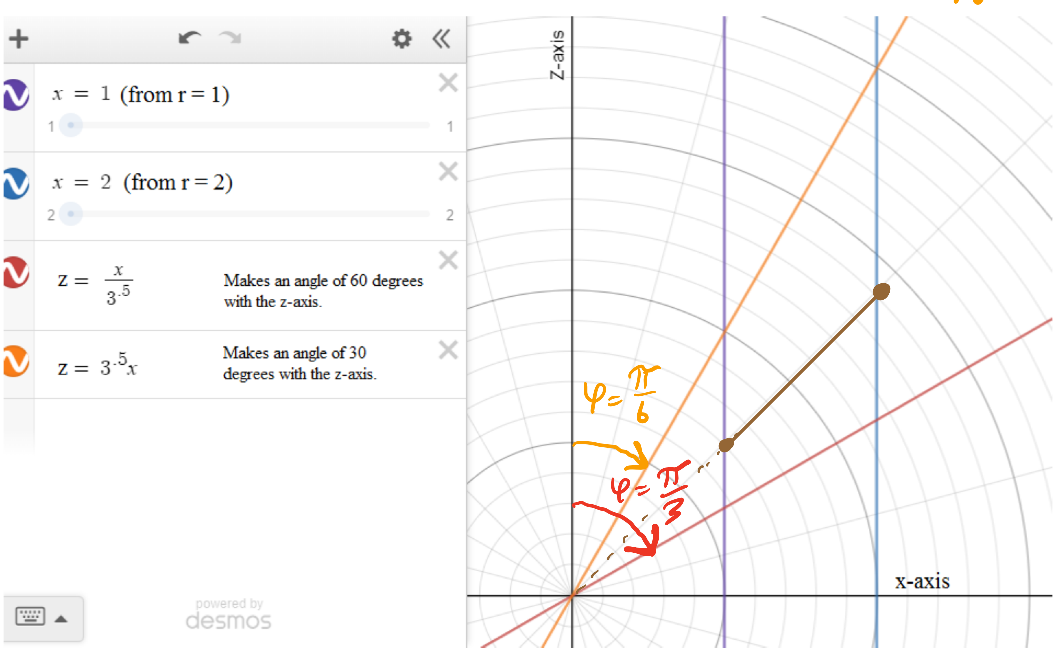
$$\mathrm{Vol}(S) = \int_0^{2\pi}\int_1^{2}\int_{\frac{r}{\sqrt{3}}}^{r\sqrt{3}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^{2}\left(rz\Big|_{z=\frac{r}{\sqrt{3}}}^{r\sqrt{3}} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{2} r^2\overbrace{\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)}^{\frac{3}{\sqrt{3}}-\frac{1}{\sqrt{3}}=\frac{2}{\sqrt{3}}} dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{2r^3}{3\sqrt{3}}\Big|_{r=1}^{2} \right) d\theta = \int_0^{2\pi} \frac{14}{3\sqrt{3}}\,d\theta$$

$$= \frac{14}{3\sqrt{3}}\cdot 2\pi = \boxed{\frac{28\pi}{3\sqrt{3}}}$$

$$\mathrm{Vol}(S) = \iiint_S 1\,dV = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$$



$\rho\sin(\varphi) = r$

$\rho\sin(\varphi) = 1$

$\rho\sin(\varphi) = 2$

$\rho = \frac{1}{\sin(\varphi)} = \csc(\varphi)$

$\rho = \frac{2}{\sin(\varphi)} = 2\csc(\varphi)$

Figure 4: The xz-plane cross section in spherical coordinates.