Problem 2: Let $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

(a) Evaluate $\iint_R \cos(x\sqrt{y})dA$.

(b) Evaluate $\iint_R x^3y\cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

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a) $\int_0^1 \underbrace{\int_0^1 \cos(x\sqrt{y})\,dx}_{\text{(y is constant here)}}\,dy$ or $\cancel{\int_0^1\int_0^1 \cos(x\sqrt{y})\,dy\,dx}$?

$u = x\sqrt{y}$

$du = \sqrt{y}\,dx$

$$= \int_0^1 \int_{x=0}^1 \frac{\cos(u)\,du}{\sqrt{y}}\,dy = \int_0^1 \frac{1}{\sqrt{y}} \int_{x=0}^1 \cos(u)\,du\,dy$$

$$= \int_0^1 \frac{1}{\sqrt{y}} \left(\sin(u) \Big|_{x=0}^1 \right) dy$$

$$= \int_0^1 \frac{1}{\sqrt{y}} \left(\sin(x\sqrt{y}) \Big|_{x=0}^1 \right) dy$$

$$= \int_0^1 \frac{1}{\sqrt{y}} \sin(\sqrt{y})\,dy \overset{\substack{u=\sqrt{y} \\ du = \frac{dy}{2\sqrt{y}}}}{=} \int_{y=0}^1 2\sin(u)\,du$$

$$= -2\cos(u)\Big|_{y=0}^{1} = -2\cos(\sqrt{y})\Big|_{y=0}^{1}$$

$$= \boxed{2-2\cos(1)}$$

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b) $\displaystyle\int_0^1\int_0^1 x^3 y\cos(x^2y^2)\,dx\,dy$ or $\displaystyle\int_0^1\int_0^1 x^3 y\cos(x^2y^2)\,dy\,dx$

c_2 c_1 c_1 $c2$

$\rightarrow u = y^2$

$\rightarrow du = 2y\,dy$

$\rightarrow$ easier option

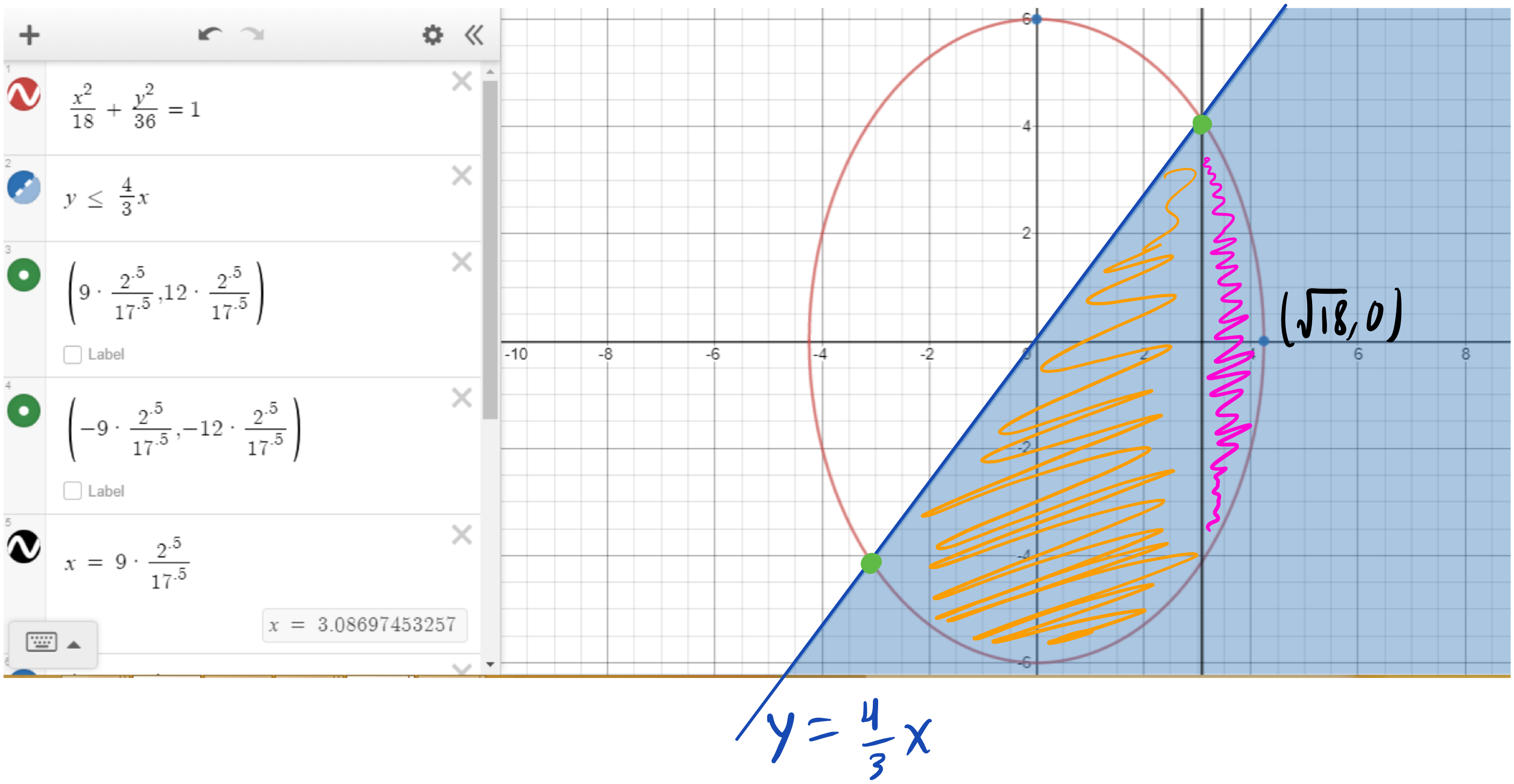
Problem 3: Let R be the region inside of the ellipse $\frac{x^2}{18}+\frac{y^2}{36}=1$ for which we also have $y \le \frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate Set up a double integral to find

$$\frac{y^2}{36}=1-\frac{x^2}{18} \rightarrow y^2 = 36-2x^2 \rightarrow y = \pm\sqrt{36-2x^2}$$

$$\iint_R xy\,dA. \qquad (2)$$



$$\frac{x^2}{18}+\frac{y^2}{36}=1 \qquad y=\frac{4}{3}x$$

$$\rightarrow 1 = \frac{x^2}{18} + \frac{\left(\frac{4}{3}x\right)^2}{36} = \frac{x^2}{18} + \frac{\frac{16}{9}x^2}{36}$$

$$= \frac{x^2}{18} + \frac{4x^2}{81} = \frac{17x^2}{162}$$

$$\rightarrow x^2 = \frac{162}{17}$$

$$\rightarrow x = \pm\sqrt{\frac{162}{17}} = \pm\frac{9\sqrt{2}}{\sqrt{17}}$$

$$(x,y) = \left(\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}\right), \left(-\frac{9\sqrt{2}}{\sqrt{17}}, \frac{-12\sqrt{2}}{\sqrt{17}}\right)$$

$$\text{area}(R) = \iint_R 1\,dA = \iint_R 1\,dy\,dx$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{\sqrt{18}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1\,dy\,dx$$

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b)

$$\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{\sqrt{18}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy\,dy\,dx$$

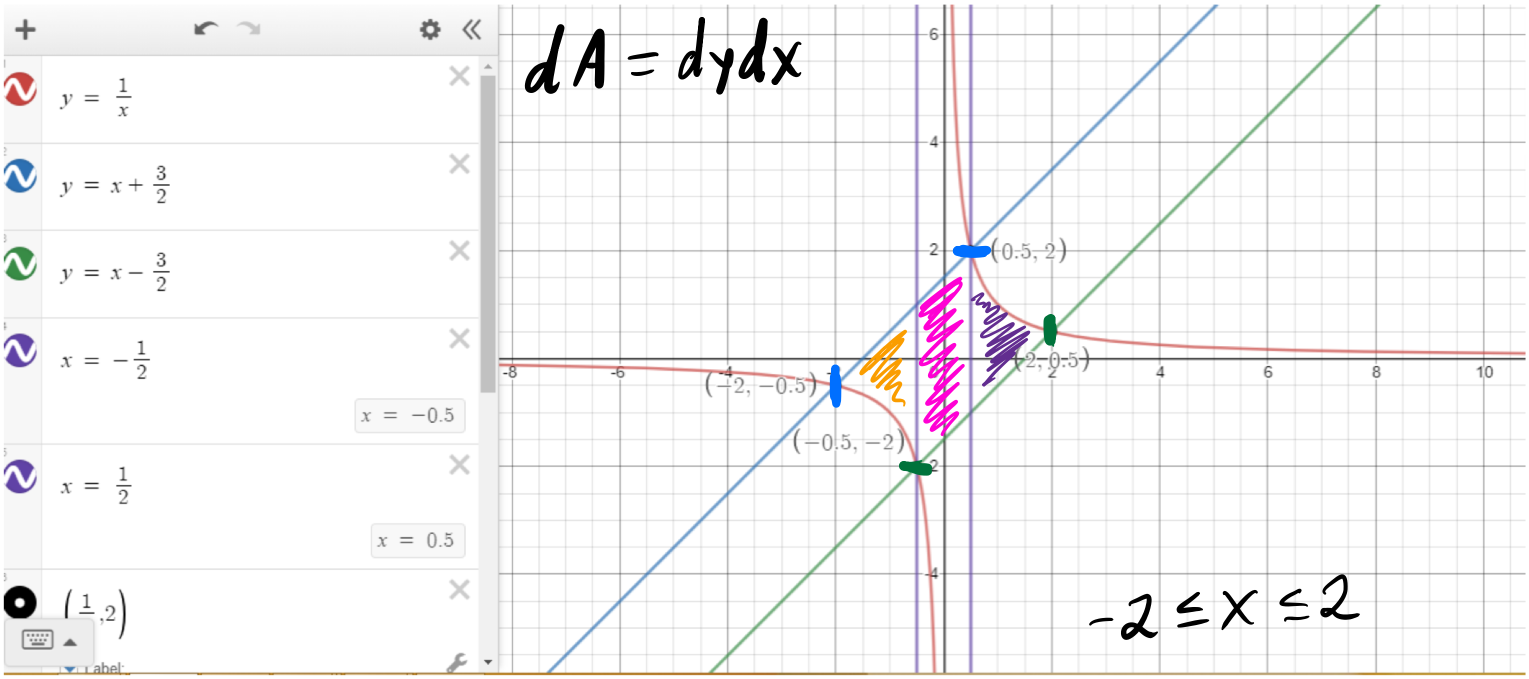
Problem 4: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.

(a) Find the area of R.

Set up a double integral to find

(b) Evaluate

$$\iint_R xy\,dA. \tag{3}$$



$$y = x + \frac{3}{2}$$
$$y = \frac{1}{x} \rightarrow \frac{1}{x} = y = x + \frac{3}{2} \rightarrow 1 = x^2 + \frac{3}{2}x$$

$$\rightarrow x^2 + \frac{3}{2}x - 1 = 0 \rightarrow \text{Q.F.} \rightarrow$$

$$x = \frac{-\frac{3}{2} \pm \sqrt{(\frac{3}{2})^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} + 4}}{2}$$

$$= \frac{-\frac{3}{2} \pm \sqrt{\frac{25}{4}}}{2} = \frac{-\frac{3}{2} \pm \frac{5}{2}}{2} = \frac{1}{2}, -2$$

$(x,y) = \left(\tfrac{1}{2}, 2\right), \left(-2, -\tfrac{1}{2}\right)$

$(x,y) = \left(-\tfrac{1}{2}, -2\right), \left(2, \tfrac{1}{2}\right)$

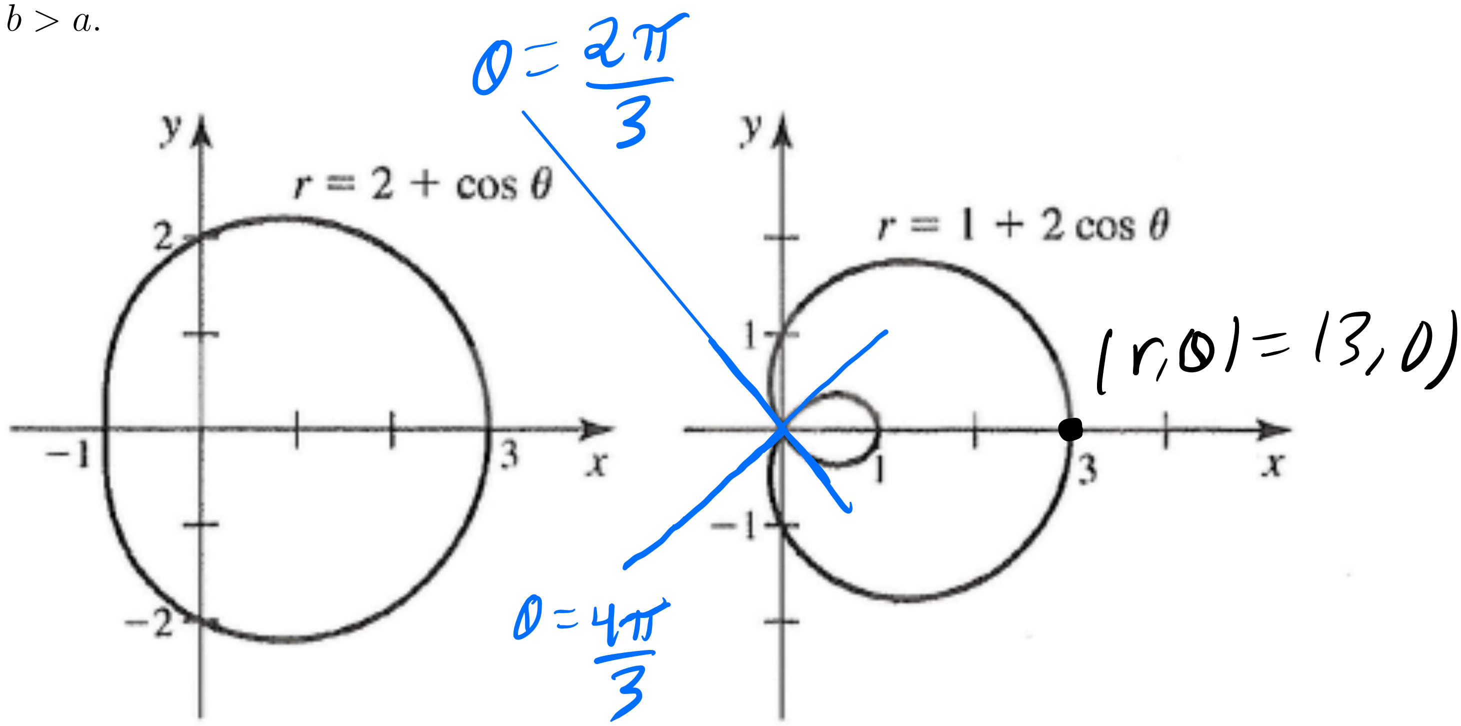
$$\text{Area}(R) = \iint_R 1\,dA = \iint_R 1\,dy\,dx$$

$$= \int_{-2}^{-\frac{1}{2}}\int_{\frac{1}{x}}^{x+\frac{3}{2}} 1\,dy\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1\,dy\,dx + \int_{\frac{1}{2}}^{2}\int_{x-\frac{3}{2}}^{\frac{1}{x}} 1\,dy\,dx$$

b

$$\int_{-2}^{-\frac{1}{2}}\int_{\frac{1}{x}}^{x+\frac{3}{2}} xy\,dy\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xy\,dy\,dx + \int_{\frac{1}{2}}^{2}\int_{x-\frac{3}{2}}^{\frac{1}{x}} xy\,dy\,dx$$

Problem 6: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.



(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.

(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.

(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

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a) $\text{Area}(R) = \iint_R 1\, dA = \iint_R r\, dr\, d\theta$

$$= \int_0^{2\pi} \int_0^{2+\cos(\theta)} r\, dr\, d\theta$$

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c) The inner loop starts and ends at the origin, which is $r=0$.

$$0 = r = 1 + 2\cos(\theta) \rightarrow \cos(\theta) = -\frac{1}{2}$$

$$\rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\rightarrow \text{area} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{1+2\cos(\theta)} r\,dr\,d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2 \Big|_{r=0}^{1+2\cos(\theta)} d\theta$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}\left(1 + 4\cos(\theta) + \underline{4\cos^2(\theta)}\right) d\theta$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = \cos^2(\theta) - \sin^2(\theta)$$
$$= 1 - 2\sin^2(\theta)$$

$$\rightarrow \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

$$= \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \left(1 + 4\cos(\theta) + \underline{2 + 2\cos(2\theta)}\right) d\theta$$

$$= \boxed{\frac{1}{2}\left(\theta + 4\sin(\theta) + 2\theta + \sin(2\theta)\right)}\Bigg|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \boxed{\pi - \frac{3}{2}\sqrt{3}}$$

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b) $\displaystyle\int_0^{2\pi}\int_0^{1+2\cos(\theta)} r\,dr\,d\theta =$

2(area inside the inner loop) +

area outside the inner loop and inside the outer loop.

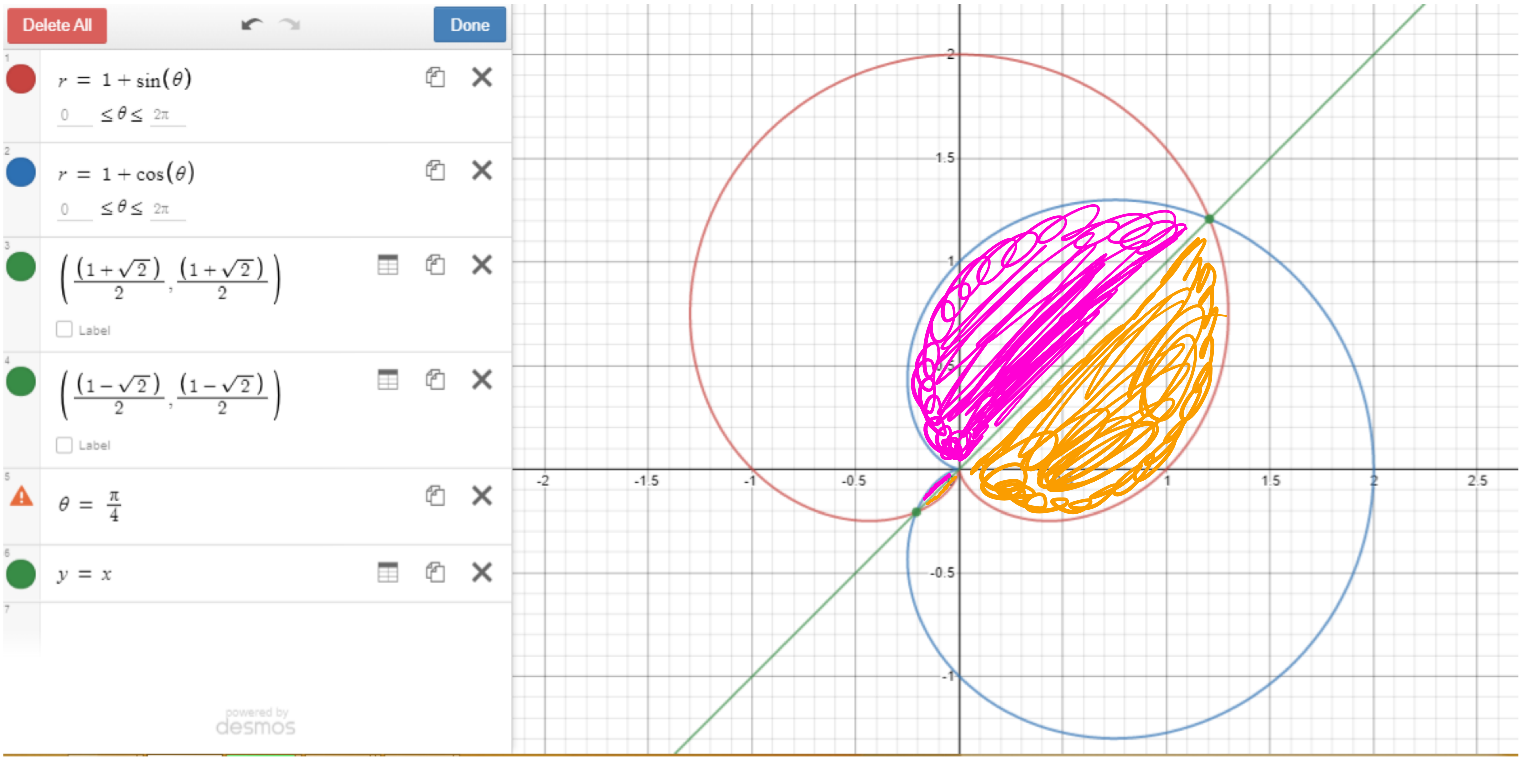
$$= \boxed{\frac{1}{2}\left(\theta + 4\sin(\theta) + 2\theta + \sin(2\theta)\right)}\Bigg|_0^{2\pi}$$

$$= 3\pi \rightarrow$$

answer $= 3\pi - 2\left(\pi - \frac{3}{2}\sqrt{3}\right)$

$$= \boxed{\pi + 3\sqrt{3}}$$

Problem 7: Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1 + \cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.



$$\text{Area}(R) = \iint_R 1\,dA = \int_{-\frac{3\pi}{4}}^{\pi/4}\int_0^{1+\sin(\theta)} r\,dr\,d\theta + \int_{\pi/4}^{5\pi/4}\int_0^{1+\cos(\theta)} r\,dr\,d\theta$$

$$1+\cos(\theta) = r = 1+\sin(\theta) \rightarrow \cos(\theta) = \sin(\theta)$$

$$\rightarrow 1 = \tan(\theta) \rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$(r,\theta) = \left(1+\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right), \underbrace{\left(1-\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)}$$

$$= \left(1-\frac{1}{\sqrt{2}}, -\frac{3\pi}{4}\right)$$

Problem 8: Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

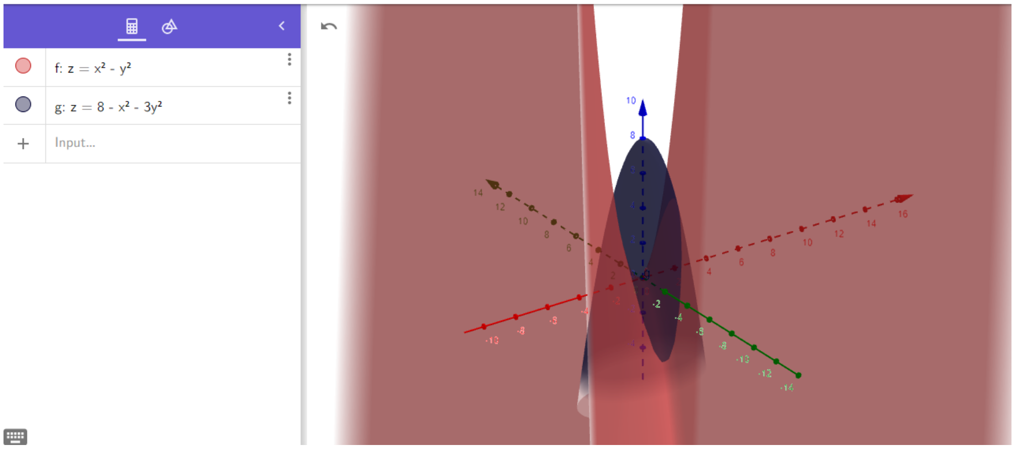


Figure 1: A view of the solid S whose volume we are calculating.

$x^2 - y^2 = z = 8 - x^2 - 3y^2 \Rightarrow 2x^2 + 2y^2 = 8$

$\rightarrow x^2 + y^2 = 4 \rightarrow$ circle of radius 2 centered at $(0,0)$.

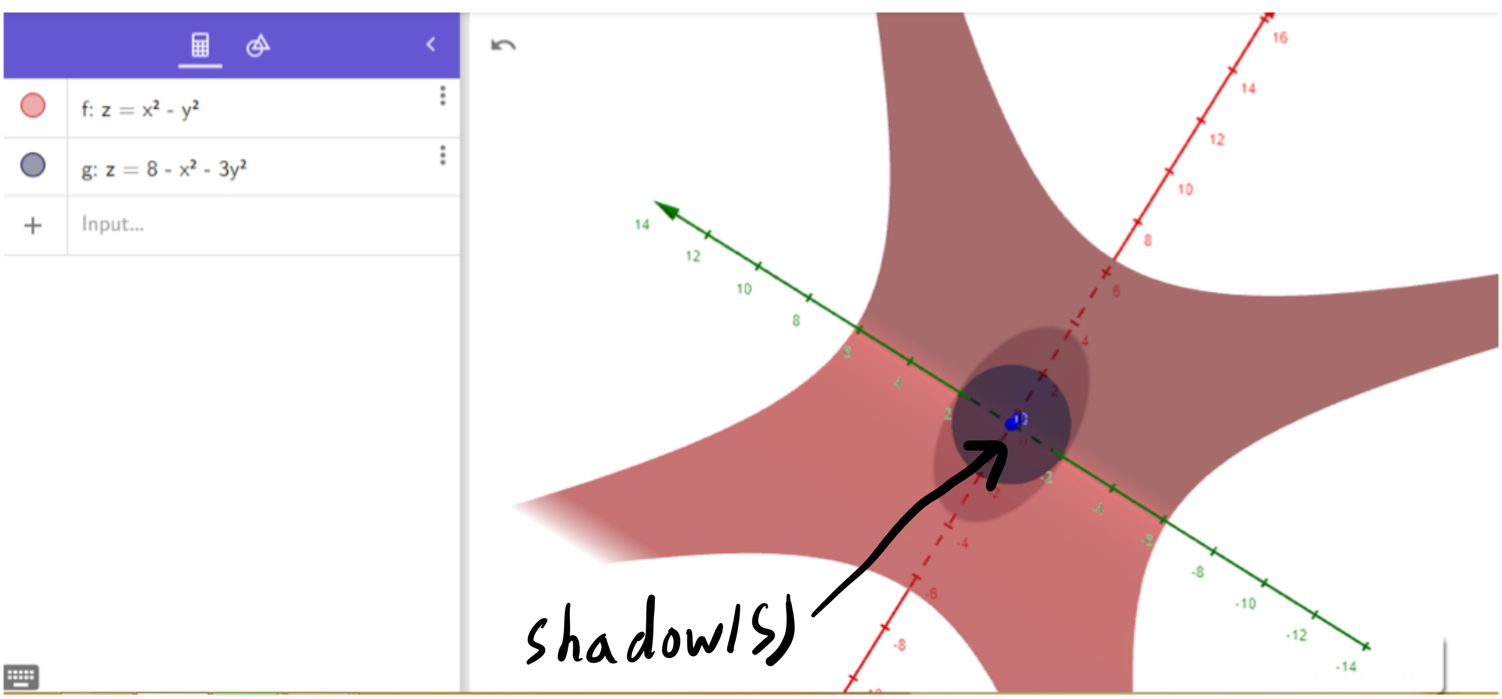


Figure 2: A bird's eye view of the solid S that is used to find the region of integration R.

$$\text{Volume}(S) = \iint_{\text{shadow}(S)} (z_{top.} - z_{bot.})\, dA$$

$$= \int_0^{2\pi} \int_0^2 \left(\boxed{z_{top.}} - \boxed{z_{bot.}}\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \Big((8 - x^2 - 3y^2) - (x^2 - y^2) \Big) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 (8 - 2x^2 - 2y^2)\, r\,dr\,d\theta$$

$x^2 + y^2 = r^2$

$$= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 (8r - 2r^3)\, dr\,d\theta$$

$$= \int_0^{2\pi} \left(4r^2 - \frac{r^4}{2} \Big|_{r=0}^{2} \right) d\theta$$

$$= \int_0^{2\pi} 8\, d\theta = 16\pi$$