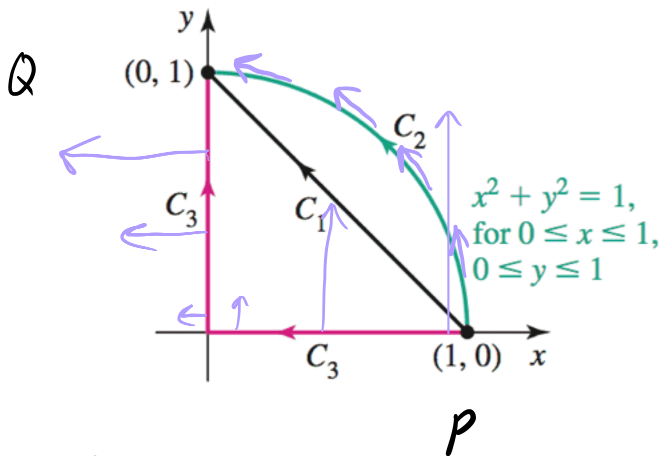
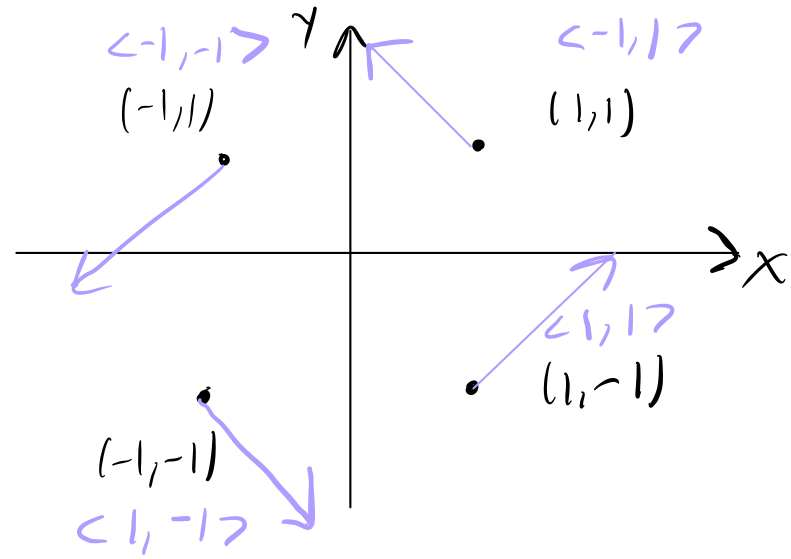
Problem 6: Consider the rotation field $\boxed{\vec{F} = \langle -y, x\rangle}$, and the three paths shown in the figure.





1. Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_1$.
2. Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_2$.
3. Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_3$.
4. Does it appear that the line integral $\int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$ is independent of the path, where $\mathcal{C}$ is any path from $(1, 0)$ to $(0, 1)$? Yes. $\vec{F}$ is not conservative.

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1) Work $= \int_{C_1} \vec{F} \cdot d\vec{r}$

Fact: a parametrization of the line segment from P to Q is given

$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}) = (1-t)\vec{P} + t\vec{Q}$, $0 \le t \le 1$.

$\vec{r}(t) = \langle 1,0\rangle + t(\langle 0,1\rangle - \langle 1,0\rangle) = \langle 1,0\rangle + t\langle -1,1\rangle$

$= \langle 1-t, t\rangle$, $0 \le t \le 1$.

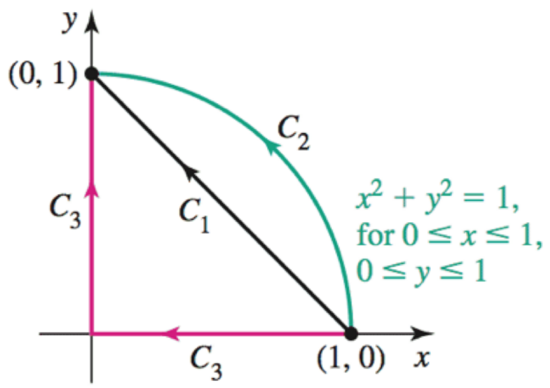
$\vec{r}'(t) = \langle -1, 1\rangle$ $\qquad \vec{F}(x,y) = \langle -y, x\rangle$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^1 \langle -t, 1-t \rangle \cdot \langle -1, 1 \rangle \, dt = \int_0^1 (t + 1 - t)\, dt$$

$$= \int_0^1 1\, dt = \boxed{1}$$

2)



$$\text{Work} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle,$$

$$0 \le t \le \frac{2\pi}{4} = \frac{\pi}{2}$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$$

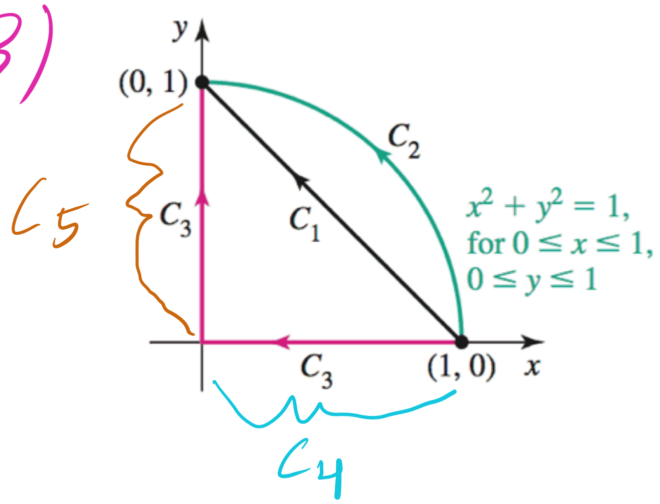
$$\vec{F}(x,y) = \langle -y, x \rangle$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2(t) + \cos^2(t))\, dt = \int_0^{\frac{\pi}{2}} 1\, dt = \boxed{\frac{\pi}{2}}$$

3)



Work $= \int_{C_3} \vec{F} \cdot d\vec{r}$

$\vec{r}_1(t) = \langle 1-t, 0 \rangle, \ 0 \le t \le 1$

is a parametrization for C_4.

$\vec{r}_1{}'(t) = \langle -1, 0 \rangle$

$\vec{r}_2(t) = \langle 0, t \rangle, \ 0 \le t \le 1$

is a parametrization for C_5.

$\vec{r}_2{}'(t) = \langle 0, 1 \rangle$

$$\vec{r}(t) = \begin{cases} \vec{r}_1(t) & \text{if } 0 \le t \le 1 \\ \vec{r}_2(t-1) & \text{if } 1 < t \le 2 \end{cases}$$

is a parametrization for C_3.

$$\vec{r}\,'(t) = \begin{cases} \vec{r}_1{}'(t) & \text{if } 0 \le t \le 1 \\ \vec{r}_2{}'(t-1) & \text{if } 1 < t \le 2 \end{cases}$$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_4} \vec{F}\cdot d\vec{r}_1 + \int_{C_5} \vec{F}\cdot d\vec{r}_2$$

$$\vec{F}(x,y) = \langle -y, x \rangle$$

$$\int_{C_4} \vec{F}\cdot d\vec{r}_1 = \int_0^1 \vec{F}(\vec{r}_1(t))\cdot \vec{r}_1\,'(t)\,dt$$

$$= \int_0^1 \langle 0, 1-t \rangle \cdot \langle -1, 0 \rangle\, dt$$

$$= \int_0^1 (0+0)\,dt = 0$$

$$\int_{C_5} \vec{F}\cdot d\vec{r}_2 = \int_0^1 \vec{F}(\vec{r}_2(t))\cdot \vec{r}_2\,'(t)\,dt$$

$$= \int_0^1 \langle -t, 0 \rangle \cdot \langle 0, 1 \rangle\, dt$$

$$= \int_0^1 (0+0)\,dt = 0.$$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_4} \vec{F}\cdot d\vec{r}_1 + \int_{C_5} \vec{F}\cdot d\vec{r}_2$$

$$= 0 + 0 = \boxed{0}$$

Problem 7: Let $f(x,y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0,0)$ and $P(1,1)$.

1. Let $\mathcal{C}_1$ be the segment from O to P. Find a parameterization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.
2. Let $\mathcal{C}_2$ be the segment from P to O. Find a parameterization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.
3. Compare the results of (1) and (2).

Fact: a curve of the form $y=f(x)$ for $a \le x \le b$ can be parametrized by $\vec{r}(t) = \langle t, f(t)\rangle$, $a \le t \le b$.

$ds = |\vec{r}\,'(t)|dt$

$\vec{r}(t) = \langle t, t^2\rangle$, $0 \le t \le 1$, $\vec{r}\,'(t) = \langle 1, 2t\rangle$

$|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$

$$\int_{C_1} f ds = \int_0^1 f(\vec{r}(t))\,|\vec{r}\,'(t)|dt = \int_0^1 t\sqrt{1+4t^2}\,dt$$

$u = 1+4t^2$

$du = 8t\,dt$

$= \ldots\ldots = \boxed{\frac{1}{12}\left(5^{\frac{3}{2}} - 1\right)}$

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2) Fact: If $\vec{r}(t)$, $a \le t \le b$ is a parametrization for a curve C, then $\vec{r}(t) = (a+b-t)$, $a \le t \le b$ is a

parametrization for C in the reverse orientation.

$(a=0, b=1)$

$\vec{r}_1(t) = \vec{r}(1-t)$, $0 \le t \le 1$ is a parametrization for C_2.

$$\int_{C_2} f\,d\vec{r}_1 = \int_0^1 f(\vec{r}_1(t))\,|\vec{r}_1'(t)|\,dt$$

$$= \int_0^1 f(\vec{r}(1-t))\,|(\vec{r}(1-t))'|\,dt$$

$$= \int_0^1 f(\vec{r}(1-t))\,|\cancel{-}\,\vec{r}\,'(1-t)|\,dt$$

$u = 1-t$

$du = -dt$

$$= \int_{u=1}^{0} f(\vec{r}(u))\,|\vec{r}\,'(u)|\,(-du)$$

$$= \int_{u=0}^{1} f(\vec{r}(u))\,|\vec{r}\,'(u)|\,du = \boxed{\frac{1}{12}\left(5^{\frac{3}{2}}-1\right)}$$

Problem 7: Let $f(x, y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

1. Let $\mathcal{C}_1$ be the segment from O to P. Find a parameterization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.
2. Let $\mathcal{C}_2$ be the segment from P to O. Find a parameterization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.
3. Compare the results of (1) and (2).

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1) Fact: $y = f(x)$ for $a \leq x \leq b$ can be parametrized by $\vec{r}(t) = \langle t, f(t) \rangle$, $a \leq t \leq b$.

so $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$ is a parametrization of C_1.

$f(x,y) = x$

$$\int_{C_1} f\,ds = \int_0^1 f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$, $|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2}$
$= \sqrt{1+4t^2}$

$$= \int_0^1 t\sqrt{1+4t^2}\,dt \underset{du = 8t}{\overset{u = 1+4t^2}{=}} \int_{t=0}^{1} \sqrt{u}\,\frac{du}{8}$$

$$= \frac{2}{3} u^{\frac{3}{2}} \cdot \frac{1}{8} \Big|_{t=0}^{1} = \frac{1}{12} (1+4t^2)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{1}{12} (1+4)^{\frac{3}{2}} - \frac{1}{12} = \boxed{\frac{1}{12} \left(5^{\frac{3}{2}} - 1 \right)}$$

Fact: If $\vec{r}(t)$, $a \le t \le b$ parametrizes a curve C, then $\vec{r}(b-t+a)$, $a \le t \le b$ parametrizes C in the opposite orientation.

$(a=0, b=1)$

$$\vec{r}_2(t) = \vec{r}(1-t) = \langle 1-t, (1-t)^2 \rangle,$$
$$0 \le t \le 1$$

is a parametrization for C_2.

$$\int_{C_2} f \, ds = \int_0^1 f(\vec{r}_2(t)) \, |\vec{r}_2{}'(t)| \, dt$$

$$= \int_0^1 f(\vec{r}(1-t))\,|(\vec{r}(1-t))'|\,dt$$

$$= \int_0^1 f(\vec{r}(1-t))\,|-\vec{r}'(1-t)|\,dt$$

$$= \int_0^1 f(\vec{r}(1-t))\,|\vec{r}'(1-t)|\,dt$$

$u = 1-t$

$du = -dt$

$$= \int_{u=1}^{0} f(\vec{r}(u))\,|\vec{r}'(u)|\,(-du)$$

$$= \int_{u=0}^{1} f(\vec{r}(u))\,|\vec{r}'(u)| = \boxed{\frac{1}{12}\left(5^{\frac{3}{2}}-1\right)}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

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3)