Problem 1: Determine all critical points of the function $f(x, y) = x^3 - y^3 + xy$, then classify each of the critical points as a local maximum, local minimum, or saddle point.

Solution: To find the criticial points of f, we simply have to find all (x, y) for which both partial derivatives of f are 0.

$$\begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 3x^2 + y & = & 0 \\ -3y^2 + x & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} -3x^2 & = & y \\ 3y^2 & = & x \end{array} \tag{1}$$

$$\rightarrow x = 3(-3x^2)^2 = 27x^4 \rightarrow x = 0, \frac{1}{3} \rightarrow (x, y) = \boxed{(0,0), (\frac{1}{3}, -\frac{1}{3})}. \tag{2}$$

We now proceed to calculate all of the second derivatives of f as well as the discriminant function so that we can apply the second derivative test.

$$\begin{array}{rcl} f_{xx}(x,y) & = & 6x \\ f_{yy}(x,y) & = & -6y \\ f_{xy}(x,y) & = & 1 \end{array} \tag{3}$$

$$\rightarrow D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 = -36xy - 1. \tag{4}$$

Since $D(0, 0) = -1 < 0$, we see that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $D(\frac{1}{3}, -\frac{1}{3}) = 3 > 0$ and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 2 > 0$ we see that $\boxed{(\frac{1}{3}, -\frac{1}{3}) \text{ is a local minimum}}$.

Problem 2: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.

Solution: If the box has a width of w, a length of ℓ and a height of h, then the volume V is given by $V = wh\ell$. We also see from figure 1 that the amount of cardboard it takes to make such a box is $2hw + 2h\ell + wl$.

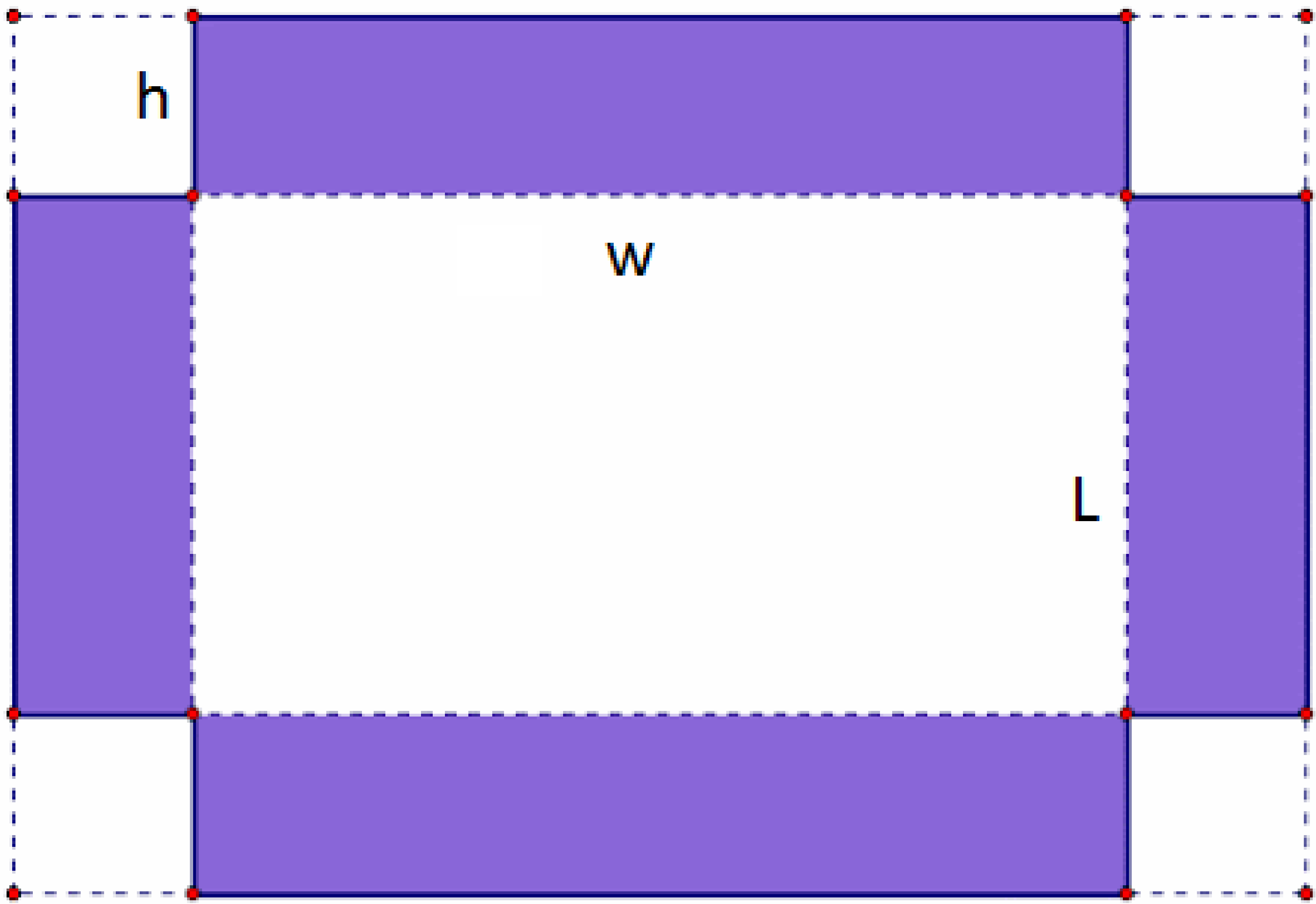


Figure 1:

It follows that we are trying to optimize the function

$$f(w, h, \ell) = 2hw + 2h\ell + w\ell \tag{5}$$

subject to the constraint

$$wh\ell = 4. \tag{6}$$

Noting that

$$h = \frac{4}{w\ell}, \tag{7}$$

we now want to optimize the function

$$g(w,\ell) = f(w,h,\ell) = f(w,\frac{4}{w\ell},\ell) = 2\frac{4}{w\ell}w + 2\frac{4}{w\ell}\ell + w\ell = \frac{8}{\ell} + \frac{8}{w} + w\ell \tag{8}$$

over the first quadrant of $\mathbb{R}^2$. We see that

$$\frac{\partial g}{\partial w} = -\frac{8}{w^2} + \ell \text{ and } \frac{\partial g}{\partial \ell} = -\frac{8}{\ell^2} + w, \text{ so} \tag{9}$$

$$\begin{matrix} \frac{\partial g}{\partial w}(w,\ell) = 0 \\ \frac{\partial g}{\partial \ell}(w,\ell) = 0 \end{matrix} \Leftrightarrow \begin{matrix} -\frac{8}{w^2} + \ell = 0 \\ -\frac{8}{\ell^2} + w = 0 \end{matrix} \Leftrightarrow 8 = w\ell^2 = w^2\ell \overset{*}{\rightarrow} w = \ell \tag{10}$$

$$\rightarrow 8 = w^3 \rightarrow (w,h,\ell) = \boxed{(2,1,2)}. \tag{11}$$

To verify that $g(w,\ell)$ at the very least attain a local minimum value at $(w,\ell) = (2,2)$ we will use the second derivative test. Technically, this step is not needed as discussed in the remark after the proof. We note that

$$\frac{\partial^2 g}{\partial w^2}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial w}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{w^2} + \ell) = \frac{16}{w^3}, \tag{12}$$

$$\frac{\partial^2 g}{\partial \ell^2}(w,\ell) = \frac{\partial}{\partial \ell}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial \ell}(-\frac{8}{\ell^2} + w) = \frac{16}{\ell^3}, \text{ and} \tag{13}$$

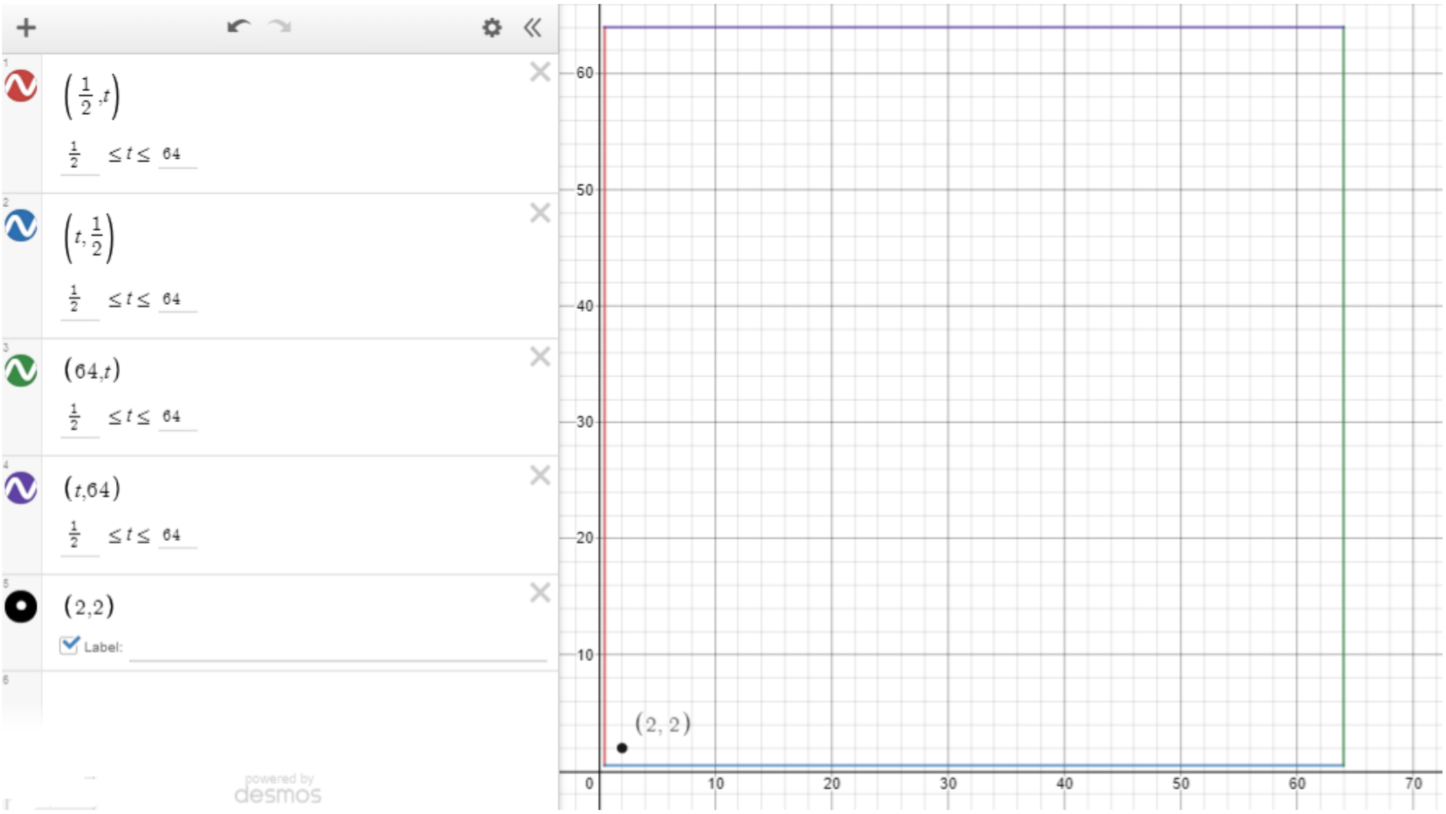
$$\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{\ell^2} + w) = 1, \text{ so} \tag{14}$$

$$D(w,\ell) = \frac{\partial^2 g}{\partial w^2}(w,\ell)\frac{\partial^2 g}{\partial \ell^2}(w,\ell) - (\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell))^2 = \frac{16}{w^3}\cdot\frac{16}{\ell^3} - 1^2 = \frac{256}{w^3\ell^3} - 1. \tag{15}$$

Since

$$D(2,2) = \frac{256}{8\cdot 8} - 1 = 3 > 0 \text{ and } \frac{\partial^2 g}{\partial w^2}(2,2) = \frac{16}{2^3} = 2 > 0, \tag{16}$$

the second derivative test tells us that $g(w,\ell)$ attains a local minimum at the critical point $(2,2)$. We will now verify that $(2,2)$ is actually the global minimum of $g(w,\ell)$ over the first quadrant of $\mathbb{R}^2$. Consider the closed and bounded region region $R = [\frac{1}{2}, 64]^2$.



A picture of R.

We note that $(2,2) \in R$, and that $(2,2)$ is the only critical point of $g(w,\ell)$ in R (because $g(w,\ell)$ only had 1 critial point anyways). We also see that $g(w,\ell) \geq 16 > 12 = g(2,2)$ for (w,ℓ) on the boundary of R (this can easily be checked on each of the 4 sides of the boundary of R separately). By the extreme value theorem, we see that g attains its absolute minimum over R at the point $(2,2)$. Since $g(w,\ell) \geq 16 > 12$ for (w,ℓ) that are in the first quadrant of $\mathbb{R}^2$ but outside of R (this fact is left as a challenge to the reader), we see that $g(w,\ell)$ does indeed attain its global minimum over the first quadrant of $\mathbb{R}^2$ at $(2,2)$.

Remark: We never actually needed to use the second derivative test to verify that the global minimum occurred at $(2,2)$. The second derivative test was only useful for telling us that $(2,2)$ was a local minimum, but we never used the fact that $(2,2)$ was a local minimum in order to conclude that it was actually a global minimum. I only wrote that into the solutions since I permitted you to finish the problem by checking that it is a local minimum instead of a global minimum. Instructors of sophomore level calculus classes usually allow for this simplification.

Problem 3: Consider the function $f(x,y) = 3 + x^4 + 3y^4$. Show that $(0,0)$ is a critical point for $f(x,y)$ and show that the second derivative test is inconclusive at $(0,0)$. Then describe the behavior of $f(x,y)$ at $(0,0)$.
Hint: The product of 2 negative numbers is positive.

Solution: We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \text{ and } \frac{\partial f}{\partial y}(x,y) = 12y^3, \text{ so} \tag{17}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3 = 0 \\ 12y^3 = 0 \end{matrix} \Leftrightarrow (x,y) = (0,0). \tag{18}$$

It follows that $(0,0)$ is the only critical point of f in all of $\mathbb{R}^2$. We also note that

$$\begin{aligned} \tfrac{\partial^2 f}{\partial x^2}(x,y) &= \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial x}(x,y) = \tfrac{\partial}{\partial x}(4x^3) = 12x^2, \\ \tfrac{\partial^2 f}{\partial y^2}(x,y) &= \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial y}(x,y) = \tfrac{\partial}{\partial y}(12y^3) = 36y^2, \text{ and} \\ \tfrac{\partial^2 f}{\partial x \partial y}(x,y) &= \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x,y) = \tfrac{\partial}{\partial x}(12y^3) = 0, \text{ so} \end{aligned} \tag{19}$$

$$\begin{aligned} D(x,y) &= \tfrac{\partial^2 f}{\partial x^2}(x,y)\tfrac{\partial^2 f}{\partial y^2}(x,y) - (\tfrac{\partial^2 f}{\partial x \partial y}(x,y))^2 \\ &= 12x^2 \cdot 36y^2 - 0^2 = 432x^2y^2 \end{aligned}. \tag{20}$$

Since $D(0,0) = 0$, we see that the second derivative test is inconclusive. However, we are still able to describe the behavior of $f(x,y)$ at $(0,0)$. Note that $x^4 \geq 0$ for all $x \in \mathbb{R}$, and $3y^4 \geq 0$ for all $y \in \mathbb{R}$. Furthermore, $x^4 = 0$ if and only if $x = 0$, and $3y^4 = 0$ if and only if $y = 0$. It follows that $x^4 + 3y^4 \geq 0$ for all $(x,y) \in \mathbb{R}^2$, and $x^4 + 3y^4 = 0$ if and only if $(x,y) = (0,0)$. From this we are able to see that $f(x,y) = 3 + x^4 + 3y^4$ attains an absolute minimum at $(0,0)$.

Problem 4: Show that the second derivative test is inconclusive when applied to the function $f(x, y) = x^4y^2$ at the point $(0, 0)$. Show that $f(x, y)$ has a local minimum at $(0, 0)$ by direct analysis.
*H*int: The product of 2 negative numbers is positive.

Solution: We will first verify that $(0, 0)$ is a critical point. We see that

$$\frac{\partial f}{\partial x}(x, y) = 4x^3y^2 \text{ and } \frac{\partial f}{\partial y}(x, y) = 2x^4y, \text{ so} \tag{21}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x, y) = 0 \\ \frac{\partial f}{\partial y}(x, y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3y^2 = 0 \\ 2x^4y = 0 \end{matrix} \Leftrightarrow x = 0 \text{ or } y = 0. \tag{22}$$

It follows that the critical points of f are precisely those points which are on either the x-axis or the y-axis, and $(0, 0)$ is certainly such a point. Next, we notice that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}(4x^3y^2) = 12x^2y^2, \\ \frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}(2x^4y) = 2x^4, \text{ and} \\ \frac{\partial^2 f}{\partial x \partial y}(x, y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial x}(2x^4y) = 8x^3y, \text{ so} \end{aligned} \tag{23}$$

$$\begin{aligned} D(x, y) &= \frac{\partial^2 f}{\partial x^2}(x, y)\frac{\partial^2 f}{\partial y^2}(x, y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right)^2 \\ &= 12x^2y^2 \cdot 2x^4 - (8x^3y)^2 = -40x^6y^2. \end{aligned} \tag{24}$$

Since $D(x, y) = 0$ whenever $x = 0$ or $y = 0$, we see that the second derivative test is inconclusive for every critical point of f (which includes (0,0)). However, we are still able to describe the behavior of $f(x, y)$ at any of its critical points by using a direct analysis. Note that $x^4y^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$ (use the hint if this is not obvious to you), and that $x^4y^2 = 0$ whenever $x = 0$ or $y = 0$. It follows that f attains its absolute minimum at any of its critical points.

Problem 5: Find the absolute minimum and absolute maximum values of the function $f(x,y) = xy$ over the region $R = \{(x,y) \mid (x-1)^2 + y^2 \leq 1\}$.

Solution: Since R is a closed and bounded region, and f is a continuous function, the Extreme Value Theorem tells us that f will attain its absolute minimum and absolute maximum values over the region R. Furthermore, we know that the extreme values of f will either be attained on the boundary of R, or at a critical point of f in the interior of R.

We will begin by finding all critical points in the interior of R. Since $f_x(x,y) = y$ and $f_y(x,y) = x$, we immediately see that $(0,0)$ is the only critical point of f, and it is on the boundary (not interior) of the region R, but it is still a candidate for where f can attain one of its extreme values. We note that $f(0,0) = 0$.

We will now proceed to find the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $\partial R = \{(x,y) \mid (x-1)^2 + y^2 = 1\}$, we will use the method of Lagrange Multipliers to optimize the function $f(x,y) = xy$ subject to the constraint $g(x,y) = (x-1)^2 + y^2 - 1 = 0$. We note that

$$\nabla f(x,y) = \langle y, x \rangle \text{ and } \nabla g(x,y) = \langle 2x-2, 2y \rangle, \tag{25}$$

so the method of Lagrange Multipliers results in the following system of equations for us to solve:

$$\begin{array}{rcl} g(x,y) & = & 0 \\ \nabla f(x,y) & = & \lambda \nabla g(x,y) \end{array} \Leftrightarrow \begin{array}{rcl} (x-1)^2 + y^2 & = & 1 \\ y & = & \lambda(2x-2) \\ x & = & \lambda 2y \end{array} \tag{26}$$

$$\to \lambda x(2x-2) = xy = \lambda 2y^2 \to 0 = 2\lambda(y^2 - x^2 + x). \tag{27}$$

By the zero-product property, we see that we must have $\lambda = 0$ or $y^2 - x^2 + x = 0$, so we will consider both cases separately.

Case 1: For our first case let us assume that $\lambda = 0$. In this case we see that the last 2 equations from (26) tell us that $x = y = 0$, since $g(0,0) = 0$, we see that we reobtain the critical point $(x,y) = (0,0)$.

Case 2: For our next case let us assume that $y^2 - x^2 + x = 0$, so $y^2 = x^2 - x$. We see that

$$1 = y^2 + (x-1)^2 = x^2 - x + (x-1)^2 = 2x^2 - 3x + 1 \tag{28}$$

$$\rightarrow 2x^2 - 3x = 0 \rightarrow x = 0, \frac{3}{2} \rightarrow (x, y) = (0, 0), (\frac{3}{2}, \frac{\sqrt{3}}{2}), (\frac{3}{2}, -\frac{\sqrt{3}}{2}). \tag{29}$$

Making a table of our critical points and corresponding values of f, we see that

(x, y)	$f(x, y)$
$(0, 0)$	0
$(\frac{3}{2}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{4}$
$(\frac{3}{2}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{4}$

so f attains its absolute maximum value of $\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ and f attains its absolute minimum value of $-\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$.

Problem 6: Find the absolute minimum and absolute maximum values of the function

$$f(x,y) = x^2 + 4y^2 + 1 \tag{30}$$

over the region

$$R = \{(x,y) : x^2 + 4y^2 \leq 1\}. \tag{31}$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

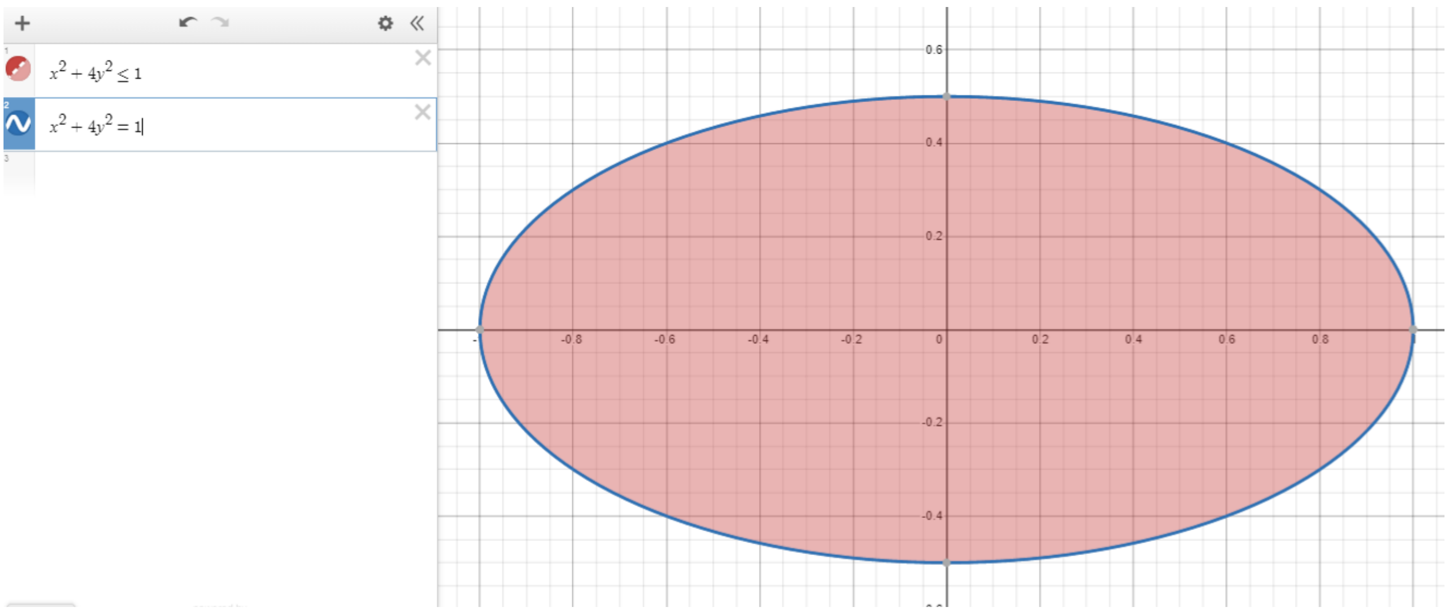


Figure 2: The interior of the R is shaded in red and the boundary of R is blue.

Solution: Since the region R is a closed and bounded region, and the function f is continuous, the extreme value theorem tells us that the absolute minimum and absolute maximum values of f must be achieved on the boundary of R or at a critical point in the interior of R. We first find all of the critical points of f. We see that

$$\begin{matrix} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{matrix} \Leftrightarrow \begin{matrix} 2x & = & 0 \\ 8y & = & 0 \end{matrix} \Leftrightarrow (x,y) = (0,0). \tag{32}$$

We see that $(0,0) \in R$ and that $f(0,0) = 1$. Next we will determine the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $x^2 + 4y^2 = 1$, we see that $f(x,y) = 2$ for every (x,y) on the boundary of R, so we immediately see that f achieves its absolute minimum value of 1 at $(0,0)$ and its absolute maximum value of 2 at any (x,y) on the boundary of R.

If we were not lucky enough to instatly notice that $f(x, y) = 2$ for every (x, y) on the boundary of R, then we would try to handle the boundary by using the method of Lagrange multipliers. More specifically, we would try to optimize the function $f(x, y) = 1 + x^2 + 4y^2$ subject to the constraint $g(x, y) = x^2 + 4y^2 - 1 = 0$. Noting that

$$\nabla g(x, y) = \langle 2x, 8y\rangle \text{ and } \nabla f(x, y) = \langle 2x, 8y\rangle \tag{33}$$

the method of Lagrange multipliers gives us the system of equations

$$\begin{array}{rcl} g(x,y) & = & 0 \\ \nabla f(x,y) & = & \lambda \nabla g(x,y) \end{array} \Leftrightarrow \begin{array}{rcl} g(x,y) & = & 0 \\ \langle 2x, 8y\rangle & = & \lambda\langle 2x, 8y\rangle \end{array} \tag{34}$$

$$\Leftrightarrow \begin{array}{rcl} g(x,y) & = & 0 \\ 2x & = & 2\lambda x \\ 8y & = & 8\lambda y \end{array}. \tag{35}$$

Letting $\lambda = 1$, we see that every point (x, y) on the boundary of R (which is the same as every point (x, y) satisfying the constraint $g(x, y) = 0$ also satisfies the system of equations given to us by the method of Lagrange multipliers. This seems bad at first since the boundary has infinitely many points, so it looks like the method of Lagrange multipliers did not help us in our search for the absolute minimum and absolute maximum values that occur on the boundary. However, it turns out that the only time every point on the boundary of our region R (assuming that R has a piecewise smooth boundary, which it always will in this class) is a critical point is when $f(x, y)$ is constant on the region R (as it was in this problem), so the problem turns out to be easier in these cases since you can determine the value of $f(x, y)$ on the boundary of R by checking the value at any random point (x, y) on the boundary of R.

Problem 7: Find the absolute minimum and maximum value of the function

$$f(x,y)=2x^2-4x+3y^2+2 \tag{36}$$

over the region

$$R:=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2\le 1\}. \tag{37}$$

Hint: There is an easy solution to this problem that doesn't use calculus if you write $f(x,y)$ in a more convenient form.

Solution: Note that the interior of R is given by

$$R^\circ=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2<1\} \tag{38}$$

and the boundary of R is given by

$$\partial R=\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+y^2=1\}. \tag{39}$$

We will first find all critical points in the interior of R. We note that

$$\frac{\partial f}{\partial x}=4x-4 \text{ and } \frac{\partial f}{\partial y}=6y, \text{ so} \tag{40}$$

$$\begin{matrix}\frac{\partial f}{\partial x}(x,y)=0\\ \frac{\partial f}{\partial y}(x,y)=0\end{matrix} \Leftrightarrow \begin{matrix}4x-4=0\\ 6y=0\end{matrix} \Leftrightarrow (x,y)=(1,0). \tag{41}$$

We see that $(1,0)$ is the only critical point of f in all of $\mathbb{R}^2$. Since $(1,0)\in R$, we have to take this critical point into consideration when searching for our absolute minimum and maximum values. Now that we have addressed the interior of R, we will proceed to address the boundary of R. We note that ∂R can be parameterized by $\vec{r}(t)$, where

$$\vec{r}(t)=(1+\cos(t),\sin(t)),\quad 0\le t\le 2\pi, \tag{42}$$

so on ∂R we have

$$\begin{aligned}f(x,y)&=f(\vec{r}(t))=f(1+\cos(t),\sin(t))\\ &=2(1+\cos(t)-1)^2+3\sin^2(t)=2\cos^2(t)+3\sin^2(t)=2+\sin^2(t).\end{aligned} \tag{43}$$

We may now use the (single variable) first derivative test to optimize $f(\vec{r}(t)) = 2+\sin^2(t)$ on the interval $[0, 2\pi]$, but we may also directly notice that the maximum is attained for $t \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ which corresponds to $(x, y) \in \{(1, 1), (1, -1)\}$ and the minimum is attained for $t \in \{0, \pi, 2\pi\}$ which corresponds to $(x, y) \in \{(0, 0), (2, 0)\}$. We now evaluate f at all of the critical points that we have found so far to determine the absolute minimum and maximum values. Noting that

(x,y)	f(x,y)
(1,0)	0
(1,1)	3
(1,-1)	3
(0,0)	2
(2,0)	2

so $f(x, y)$ attains a minimum value of 0 at $(1, 0)$, and $f(x, y)$ attains a maximum value of 3 at any of $\{(1, 1), (1, -1)\}$.

Remark: In this problem, one may also try to address the boundary of R by noting that $(x-1)^2 = 1 - y^2$ on the boundary, so $f(x, y) = 2(x-1)^2 + 3y^2 = 2 + y^2$ on the boundary.

Problem 8: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

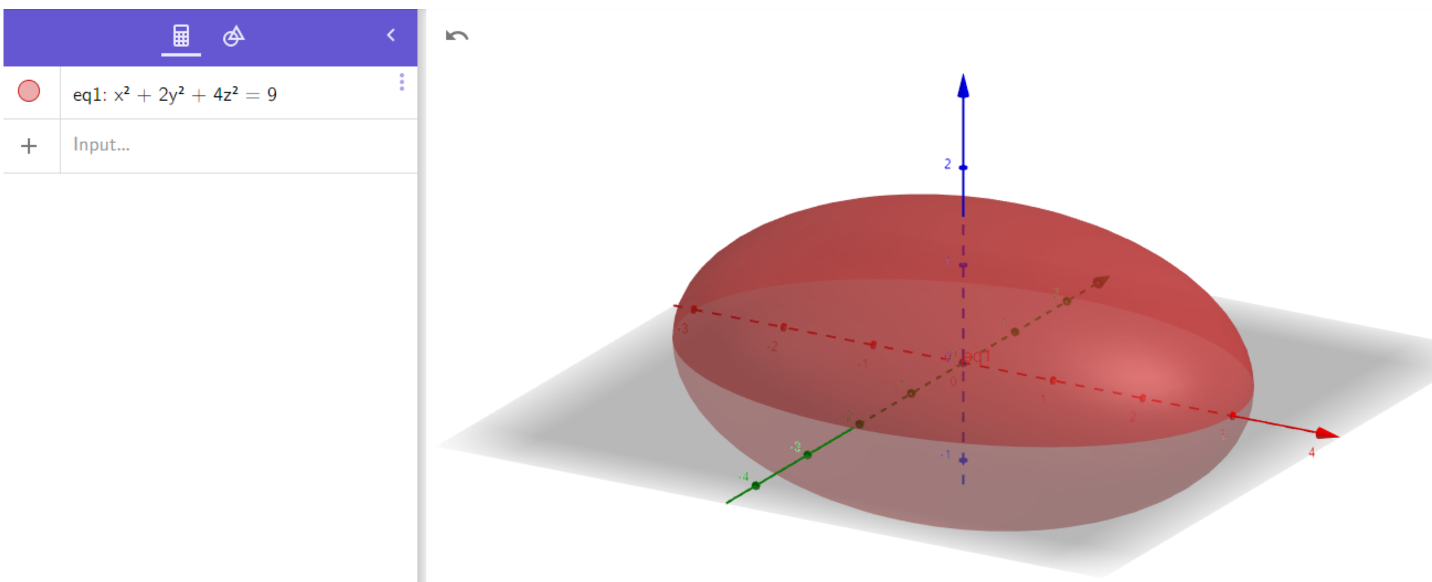
$$f(x, y, z) = xyz \tag{44}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{45}$$

Solution: We will present two different solutions to this problem. The method of setting up the system of equations from the method of Lagrange multipliers is the same in both solutions, but the method of solving the resulting system will be different.

We see that the region defined by the constraint is a closed and bounded region with no boundary, so the method of Lagrange multipliers will give us the complete list of critical points that we need to check in order to determine the absolute minimum and absolute maximum values of f subject to the constraint.



We see that

$$x^2 + 2y^2 + 4z^2 = 9 \Leftrightarrow x^2 + 2y^2 + 4z^2 - 9 = 0, \tag{46}$$

so we may take our constraint function to be $g(x, y, z) = x^2 + 2y^2 + 4z^2 - 9$. We see that

$$\vec{\nabla} f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle = \langle yz, xz, xy\rangle, \text{ and} \tag{47}$$

$$\vec{\nabla} g(x, y, z) = \langle g_x(x, y, z), g_y(x, y, z), g_z(x, y, z) = \langle 2x, 4y, 8z\rangle. \tag{48}$$

We now want to find all (x, y, z, λ) (although we don't really care about the value of λ) such that

$$\begin{aligned} g(x,y,z) &= 0 \\ \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \end{aligned} \tag{49}$$

. . . .

$$\Leftrightarrow \begin{aligned} x^2 + 2y^2 + 4z^2 - 9 &= 0 \\ \langle yz, xz, xy \rangle &= \lambda \langle 2x, 4y, 8z \rangle \end{aligned} \tag{50}$$

. . . .

$$\Leftrightarrow \begin{aligned} x^2 + 2y^2 + 4z^2 - 9 &= 0 \\ yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= 8\lambda z \end{aligned} \tag{51}$$

Finish 1: We will now use the method of cross multiplication to solve the system of equations in (51). This method will be computationally intensive, but is 'standard' and does not require any 'tricky insights'. By cross multiplying the second and third equations in (51) we see that

$$4\lambda y^2 z = 2\lambda x^2 z \to 0 = 4\lambda y^2 z - 2\lambda x^2 z = 2\lambda z(2y^2 - x^2), \tag{52}$$

so by the zero product property we see that either $\lambda = 0, z = 0$, or $2y^2 - x^2 = 0$. We will handle each case separately.

Case 1 ($\lambda = 0$): By plugging $\lambda = 0$ back into (51) we see that

$$\begin{aligned} x^2 + 2y^2 + 4z^2 - 9 &= 0 \\ yz &= 0 \\ xz &= 0 \\ xy &= 0 \end{aligned} . \tag{53}$$

Using the zero product property once again on the second, third, and fourth equations of (53), we see that 2 of x, y, and z must be 0. In conjunction with the first equation of (51) (the constraint equation) we see that $(x, y, z, \lambda) \in \{(0, 0, \pm\frac{3}{2}, 0), (0, \pm\frac{3}{\sqrt{2}}, 0, 0), (\pm 3, 0, 0, 0)\}$ are the solutions that we obtain from this case.

Case 2 (z = 0): By plugging $z = 0$ back into (51) we see that

$$\begin{array}{rcl} x^2+2y^2-9 & = & 0 \\ 0 & = & 2\lambda x \\ 0 & = & 4\lambda y \\ xy & = & 0 \end{array}. \tag{54}$$

Since we are done with case 1, we may also assume that $\lambda \neq 0$. It now follows from the second and third equations in (54) that $x = y = 0$, but this contradicts the first equation in (54), so we obtain no additional solutions in this case.

Case 3 ($2y^2 - x^2 = 0$): In this case we see that $x^2 = 2y^2$ so $x = \pm\sqrt{2}y$, which means that we have 2 subcases to handle. For our first subcase, we plug $x = \sqrt{2}y$ back into (51) to obtain

$$\begin{array}{rcl} 2y^2+2y^2+4z^2-9 & = & 0 \\ yz & = & 2\sqrt{2}\lambda y \\ \sqrt{2}yz & = & 4\lambda y \\ \sqrt{2}y^2 & = & 8\lambda z \end{array}. \tag{55}$$

By cross-multiplying the third and fourth equations in (55) we see that

$$8\sqrt{2}\lambda yz^2 = 4\sqrt{2}\lambda y^3 \to 0 = 8\sqrt{2}\lambda yz^2 - 4\sqrt{2}\lambda y^3 = 4\sqrt{2}\lambda y(2z^2 - y^2). \tag{56}$$

Since we are no longer in case 1, we may assume that $\lambda \neq 0$, so either $y = 0$ or $2z^2 - y^2 = 0$. If $y = 0$, then $x = \sqrt{2}y = 0$, and we reobtain the solution $(x, y, z) = (0, 0, \frac{3}{2})$. If $2z^2 - y^2 = 0$, then $y^2 = 2z^2$. Plugging this back into the first equation of (55) yields

$$12z^2 = 9 \to z = \pm\frac{\sqrt{3}}{2}, \tag{57}$$

so we obtain the solutions

$$(x,y,z) \in \{(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{58}$$

For our second subcase we let $x = -\sqrt{2}y$ and a similar calculation yields the additional solutions

$$(x,y,z) \in \{(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}),$$
$$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \quad (59)$$

Now that we have found all solutions to the system of equations in (51), we see that

(x,y,z)	f(x,y,z)		(x,y,z)	f(x,y,z)
(0,0,$\frac{3}{2}$)	0		$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(0,$\frac{3}{\sqrt{2}}$,0)	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(3,0,0)	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(0,0,$-\frac{3}{2}$)	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(0,$-\frac{3}{\sqrt{2}}$,0)	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(-3,0,0)	0		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the absolute minimum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the absolute maximum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.

. . . .

Finish 2: We will now use the symmetry that appears in the system of equations in (51) in order to solve the system more quickly. Observe that

$$\begin{array}{rcl} x^2+2y^2+4z^2-9 & = & 0 \\ yz & = & 2\lambda x \\ xz & = & 4\lambda y \\ xy & = & 8\lambda z \end{array} \rightarrow \begin{array}{rcl} x^2+2y^2+4z^2-9 & = & 0 \\ xyz & = & 2\lambda x^2 \\ xyz & = & 4\lambda y^2 \\ xyz & = & 8\lambda z^2 \end{array} \quad (60)$$

$$\rightarrow \lambda x^2 = 2\lambda y^2 = 4\lambda z^2. \quad (61)$$

We now have 2 cases to consider based on whether or not $\lambda = 0$.

Case 1 ($\lambda = 0$): In this case, we plug $\lambda = 0$ into the system of equations appearing in the left hand portion of (60) (the original system of equations that we started with) to see that

$$\begin{aligned} x^2 + 2y^2 + 4z^2 - 9 &= 0 \\ yz &= 0 \\ xz &= 0 \\ xy &= 0 \end{aligned} \rightarrow (x, y, z) \in \{(x, 0, 0), (0, y, 0), (0, 0, z)\}. \quad (62)$$

$$\rightarrow (x, y, z) \in \{(\pm 3, 0, 0), (0, \pm\frac{3}{\sqrt{2}}, 0), (0, 0, \pm\frac{3}{2})\}. \quad (63)$$

Case 2 ($\lambda \neq 0$): In this case, we see that we can divide the equations appearing in (61) by λ and plug to result back into our constraint equation to obtain

$$x^2 = 2y^2 = 4z^2 \rightarrow 9 = x^2 + 2y^2 + 4z^2 = 3x^2 \rightarrow x = \pm\sqrt{3}, \text{ and} \quad (64)$$

$$(x, y, z) \in \{(x, \frac{x}{\sqrt{2}}, \frac{x}{2}), (x, -\frac{x}{\sqrt{2}}, \frac{x}{2}), (x, \frac{x}{\sqrt{2}}, -\frac{x}{2}), (x, -\frac{x}{\sqrt{2}}, -\frac{x}{2})\}. \quad (65)$$

Putting together all of our results from cases 1 and 2, we once again find all solutions to the system of equations in (60) as

(x,y,z)	f(x,y,z)		(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0		$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,\frac{3}{\sqrt{2}},0)$	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(3,0,0)	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,0,-\frac{3}{2})$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,-\frac{3}{\sqrt{2}},0)$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(-3,0,0)	0		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the absolute minimum value of $f(x, y, z)$ subject to $g(x, y, z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the absolute maximum value of $f(x, y, z)$ subject to $g(x, y, z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.

Problem 9: What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.

Solution: Note that for any (x, y, z) on the plane $x + y + 4z = 8$ we have

$$z = 2 - \frac{1}{4}x - \frac{1}{4}y, \tag{66}$$

from which we see that

$$\mathrm{d}((x, y, z), (0, 0, 0)) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \tag{67}$$

$$= \sqrt{x^2 + y^2 + (2 - \frac{1}{4}x - \frac{1}{4}y)^2} = \sqrt{4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2}. \tag{68}$$

We recall that if $f(x, y)$ is any nonnegative function, then $f(x, y)$ and $f^2(x, y)$ have their (local and global) minimums and maximums occur at the same values of (x, y). It follows that we want to optimize the function

$$f(x, y) = 4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2. \tag{69}$$

Since any global minimum of $f(x, y)$ is also a local minimum, we see that the global minimum of f (if it exists) is at a critical point. We now begin finding the critical points of f. We see that

$$\begin{matrix} 0 = f_x(x, y) = \frac{17}{8}x + \frac{1}{8}y - 1 \\ 0 = f_y(x, y) = \frac{17}{8}y + \frac{1}{8}x - 1 \end{matrix} \to 0 = (\frac{17}{8}x + \frac{1}{8}y - 1) - (\frac{17}{8}y + \frac{1}{8}x - 1) \tag{70}$$

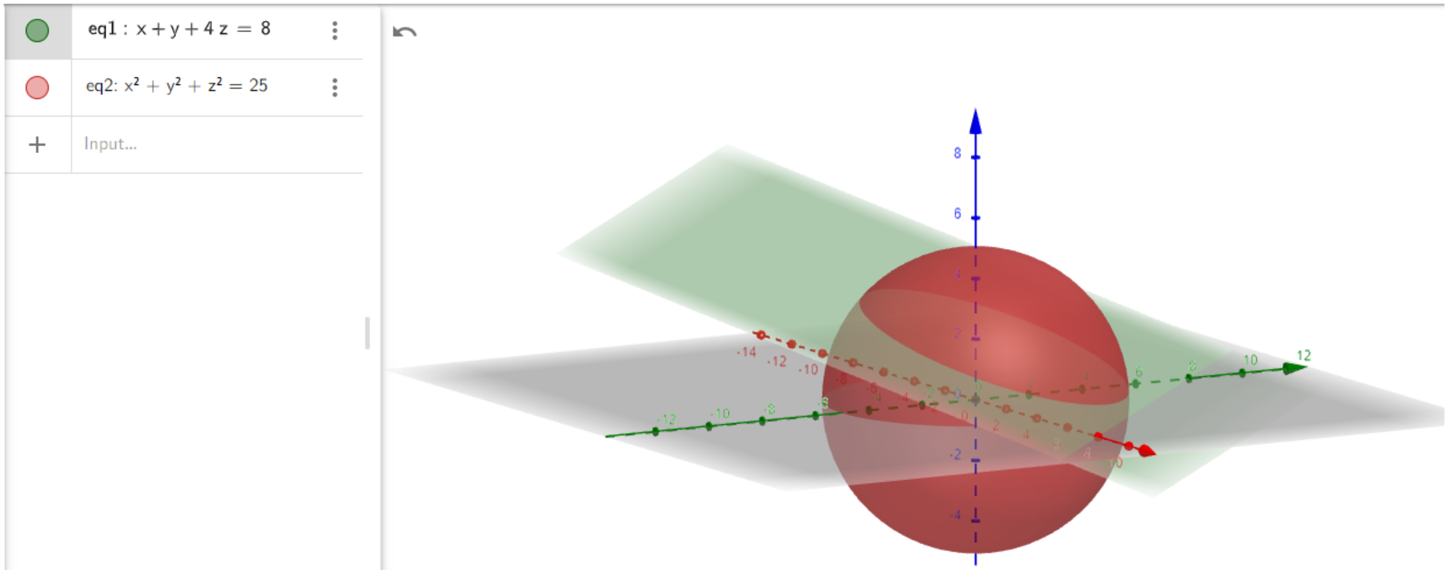
$$= 2x - 2y \to x = y \to x = y = \frac{4}{9}. \tag{71}$$

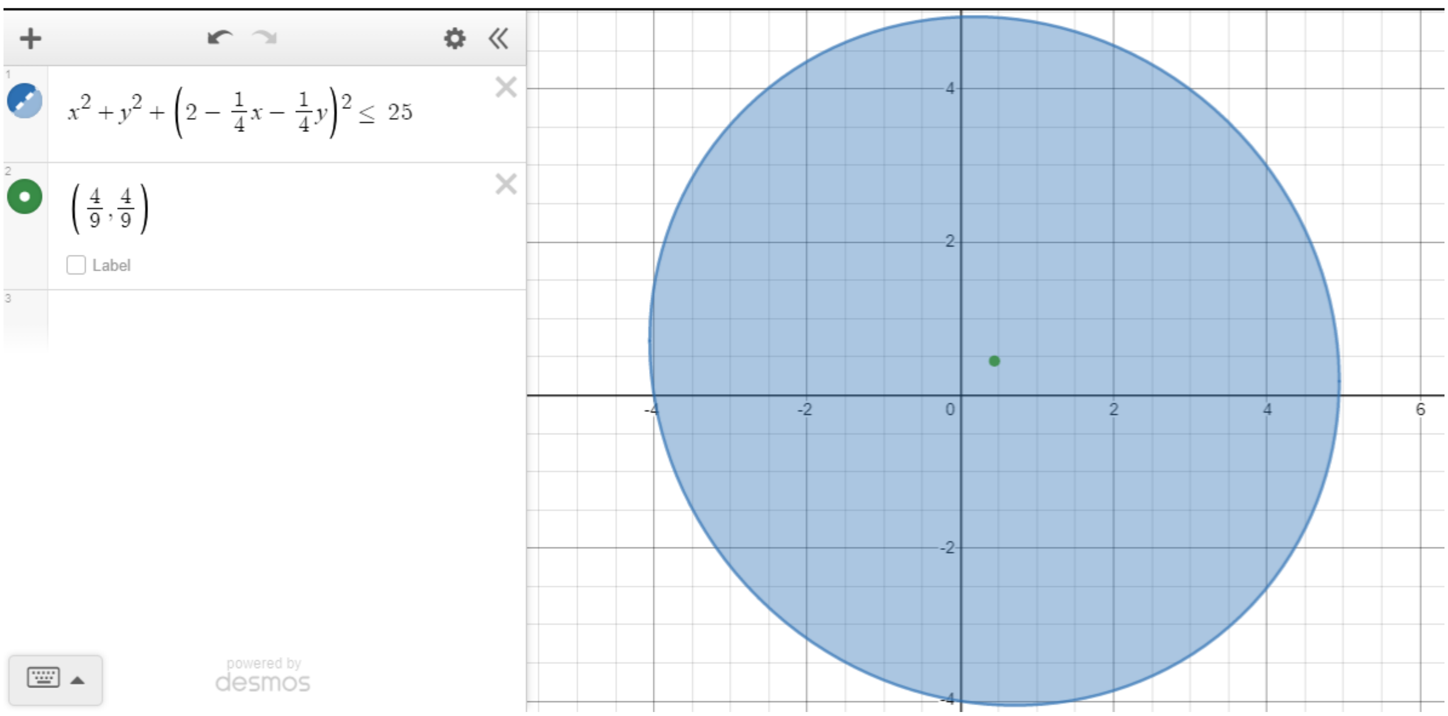
We see that $(\frac{4}{9}, \frac{4}{9})$ is the only critical point. We will now use the second derivative test to verify that $(\frac{4}{9}, \frac{4}{9})$ is a local minimum. We see that

$$\begin{matrix} f_{xx}(x, y) = \frac{17}{8} \\ f_{yy}(x, y) = \frac{17}{8} \\ f_{xy}(x, y) = \frac{1}{8} \end{matrix} \to D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 \tag{72}$$

$$= \frac{17}{8} \cdot \frac{17}{8} - (\frac{1}{8})^2 = \frac{9}{2} \to D(\frac{4}{9}, \frac{4}{9}) = \frac{9}{2} > 0. \tag{73}$$

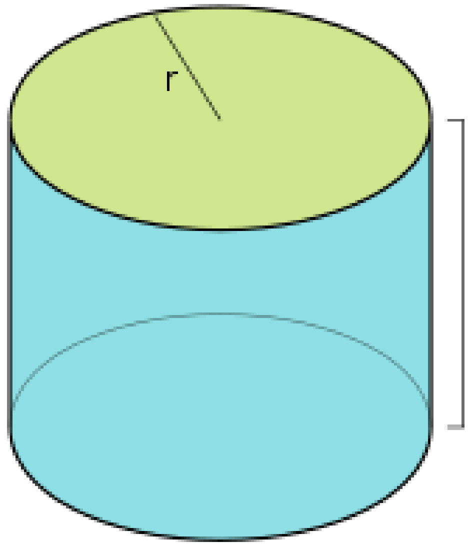
Since we also see that $f_{xx}(\frac{4}{9},\frac{4}{9}) = \frac{17}{8} > 0$, the second derivative test tells us that $(\frac{4}{9},\frac{4}{9})$ is indeed a local minimum of $f(x,y)$. It remains to show that $f(x,y)$ attains its global minimum at $(\frac{4}{9},\frac{4}{9})$. Firstly, we note that $f(\frac{4}{9},\frac{4}{9}) = \frac{32}{9}$. Since $\frac{32}{9} < 25$ (I picked 25 randomly, I just needed some larger number), let us consider the region R of (x,y) for which $(x,y,\underbrace{2-\frac{1}{4}x-\frac{1}{4}y}_{z})$ has a distance of at most 5 from the origin. This is the same as $R = \{(x,y) \mid f(x,y) \le 25\}$.





Since R is a closed and bounded region, and $f(x,y)$ is a continuous function function, we know that f attains an absolute minimum on R. The point $(\frac{4}{9},\frac{4}{9})$ is inside of R, so the minimum of f is not attained on the boundary of R (as that is where the distance to the origin is exactly 5). Since the minimum of f on R is attained on the interior, we see that it must be obtained at a critical point of $f(x,y)$, so it is attained at $(\frac{4}{9},\frac{4}{9})$. For any point (x,y) outside of R, we have $f(x,y) > 25$ (by the very definition of R), so the global minimum of $f(x,y)$ is $\frac{32}{9}$ and is attained at $(\frac{4}{9},\frac{4}{9})$. It follows that the point on the plane $x+y+4z=8$ that is closest to the origin is $\boxed{(\frac{4}{9},\frac{4}{9},\frac{16}{9})}$.

Problem 10: Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.



Solution: We recall that a cylinder of radius r and height h has a volume of $V = \pi r^2 h$ and a surface area (including the 2 circular ends) of $S = 2\pi r^2 + 2\pi rh$. It follows that we want to optimize the function $f(r,h) = 2\pi r^2 + 2\pi rh$ subject to the constraint $0 = g(r,h) = \pi r^2 h - 32\pi$. Since

$$\nabla f(r,h) = \langle 4\pi r + 2\pi h, 2\pi r\rangle \text{ and } \nabla g(r,h) = \langle 2\pi rh, \pi r^2\rangle, \text{ we obtain} \tag{74}$$

$$\begin{matrix} 4\pi r + 2\pi h & = & 2\pi\lambda rh & & 2r + h & = & \lambda rh & & 2r + h & = & 2h \\ 2\pi r & = & \pi\lambda r^2 & \overset{r\neq 0}{\to} & 2 & = & \lambda r & \to & 2 & = & \lambda r \\ \pi r^2 h & = & 32\pi & & r^2 h & = & 32 & & r^2 h & = & 32 \end{matrix} \tag{75}$$

$$\begin{matrix} & 2r & = & h & & 2r & = & h \\ \to & 2 & = & \lambda r & \to & 2 & = & \lambda r & \to r = \sqrt[3]{16} = 2\sqrt[3]{2} \to h = 4\sqrt[3]{2}. \\ & r^2 h & = & 32 & & 2r^3 & = & 32 \end{matrix} \tag{76}$$

Since the cylinder does not have a maximum surface area when subjected to the constraint $V = 32\pi$, we see that the critical point that we found has to correspond to a local minimum. The extreme/boundary cases occur when either $r \to \infty$ or $h \to \infty$, in which case we also have $S \to \infty$. It follows that $f(r,h)$ attains a minimum value of $24\pi\sqrt[3]{4}$ when $(r,h) = \boxed{(2\sqrt[3]{2}, 4\sqrt[3]{2})}$.

Problem 11: Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K,L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the production function

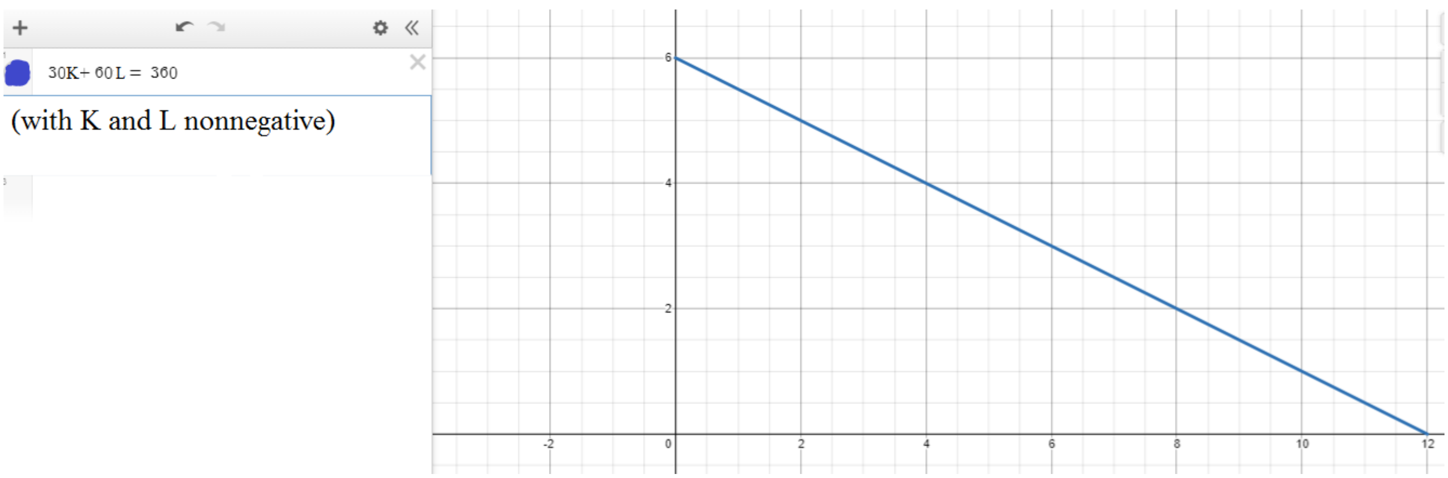
$$P = f(K,L) = 10K^{\frac{1}{3}}L^{\frac{2}{3}} \tag{77}$$

subject to

$$30K + 60L = 360, \tag{78}$$

assuming $K \geq 0$ and $L \geq 0$.

Solution: We see that the region defined by the constraint is the line segment from $(K,L) = (0,6)$ to $(K,L) = (12,0)$, which is a closed and bounded region with boundary.



The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K,L)$, its gradient field $\nabla g(K,L)$, and the gradient field $\nabla f(K,L)$ of our optimization function as

$$g(K,L) = 30K + 60L - 360, \nabla g(K,L) = \langle 30, 60\rangle, \text{ and} \tag{79}$$

$$\nabla f(K,L) = \langle \frac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}\rangle. \tag{80}$$

The method of Lagrange multipliers gives us the system of equations

$$\begin{aligned} g(K,L) &= 0 \\ \nabla f(K,L) &= \lambda \nabla g(K,L) \end{aligned} \tag{81}$$

$$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \langle \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} \rangle &= \lambda\langle 30, 60\rangle \end{aligned} \tag{82}$$

$$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} &= 30\lambda \\ \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} &= 60\lambda \end{aligned} \tag{83}$$

$$\to \frac{20}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} = 60\lambda = \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} \to K = L \tag{84}$$

$$\to 0 = 30K + 60L - 360 = 90L - 360 \to L = 4 \to \boxed{(K,L) = (4,4)}. \tag{85}$$

Since $(4,4)$ is the only critical point given to use by the method of Lagrange multipliers and

$$f(4,4) = 10 \cdot 4^{\frac{1}{3}}4^{\frac{2}{3}} = 10 \cdot 4 = 40 > 0 = f(12,0) = f(0,6), \tag{86}$$

we see that the production function attains its absolute maximum value (subject to the given constraint) of 40 at $(4,4)$.

Problem 12: Given the production function $P = f(K, L) = K^aL^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1-a)B/q$. (Recall that $p, q, K, L \geq 0$ and $0 < a < 1$ in order for the model to make sense in the real world and for the production function f to be well defined.)

Solution: We see that the region defined by the constraint is the line segment from $(K, L) = (0, \frac{B}{q})$ to $(K, L) = (\frac{B}{p}, 0)$, which is a closed and bounded region with boundary. The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K, L)$, its gradient field $\nabla g(K, L)$, and the gradient field $\nabla f(K, L)$ of our optimization function as

$$g(K, L) = pK + qL - B, \nabla g(K, L) = \langle p, q\rangle, \text{ and} \tag{87}$$

$$\nabla f(K, L) = \langle aK^{a-1}L1 - a, (1-a)K^aL^{-a}\rangle. \tag{88}$$

The method of Langrange multipliers gives us the system of equations

$$\begin{aligned} g(K, L) &= 0 \\ \nabla f(K, L) &= \lambda \nabla g(K, L) \end{aligned} \tag{89}$$

$$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ \langle aK^{a-1}L1 - a, (1-a)K^aL^{-a}\rangle &= \lambda\langle p, q\rangle \end{aligned} \tag{90}$$

$$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ aK^{a-1}L1 - a &= p\lambda \\ (1-a)K^aL^{-a} &= q\lambda \end{aligned} \tag{91}$$

$$\to qaK^{a-1}L^{1-a} = pq\lambda = p(1-a)K^aL^{-a} \tag{92}$$

$$\to qaL = p(1-a)K \to L = \frac{p(1-a)}{qa}K \tag{93}$$

$$\to 0 = pK + qL - B = pK + \frac{p(1-a)}{a}K - B \to K = \frac{Ba}{p} \tag{94}$$

. . . .

$$\to L \overset{\text{(By (93))}}{=} \frac{B(1-a)}{q}, \text{ so} \tag{95}$$

. . . .

$$(K, L) = (\frac{Ba}{p}, \frac{B(1-a)}{q}) \tag{96}$$

is the only critical point obtained by the method of Lagrange multipliers. We see that $K, L > 0$ at this critical point, so

$$f(K, L) > 0 = f(0, \frac{B}{q}) = f(\frac{B}{p}, 0). \tag{97}$$

Since the value of f at the (only) critical point is larger than the values of f on the boundary (the end points) we see that f attains its absolute maximum value at the critical point as desired.