Problem 1: Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.

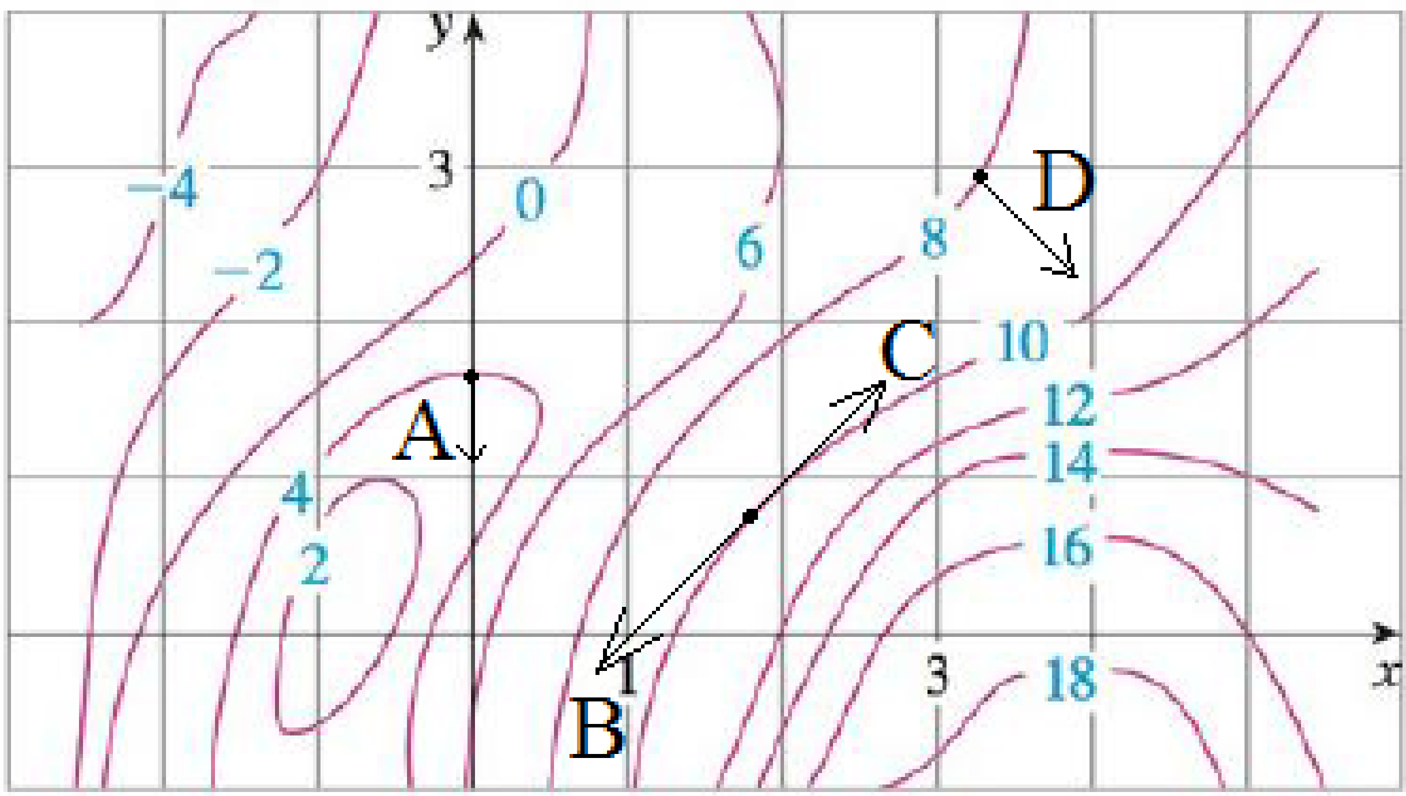


Figure 1: Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) (D) (E) None of the given vectors

Explanation: The gradient vector of a function $f(x, y)$ is normal to the level curves (the curves of the form $f(x, y) = c$, with c a constant) and points in the direction of maximum increase. We see that vector A is normal to a level curve of f, but points in the direction of decrease and is therefore not a gradient vector. We see that vectors B and C are tangent to a level curve, not normal to the level curve, so neither of them can be a gradient vector. We see that vector D is normal to a level curve of f and points in the direction of increase, so D could be a gradient vector of f.

Problem 2: Consider the function $f(x,y) = x^2 + y^2$ and and the point $P = (2,3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.

Solution to (a): We see that $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2x, 2y\rangle$. We see that $-\nabla f(2,3) = \langle -4, -6\rangle$ is a vector that points in the direction of maximum decrease of f at the point P. Since $|\langle -4, -6\rangle| = \sqrt{52} = 2\sqrt{13}$, we see that

$$\frac{\langle -4, -6\rangle}{|\langle -4, -6\rangle|} = \frac{1}{2\sqrt{13}}\langle -4, -6\rangle = \boxed{\langle \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\rangle} \tag{1}$$

is the direction of maximum decrease of f at the point P.

Solution to (b): We see that $|\vec{u}| = \sqrt{13}$, so

$$\vec{w} = \frac{\vec{u}}{|\vec{u}|} = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \tag{2}$$

is the unit vector that points in the same direction as $\vec{u}$, so

$$d_{\vec{w}} f(2,3) = \nabla f(2,3)\cdot \vec{w} = \langle 4, 6\rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle = \boxed{\frac{24}{\sqrt{13}}}. \tag{3}$$

Problem 3: Consider the function $f(x,y) = \ln(1+4x^2+3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

(a) Find the gradient field $\nabla f(x,y)$ of $f(x,y)$ and then evaluate it at P.

(b) Find the angles θ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

(c) Write the directional derivative at P as a function of θ; call this function $g(\theta)$.

(d) Find the value of θ that maximizes $g(\theta)$ and find the maximum value.

(e) Verify that the value of θ that maximizes g corresponds to the direction of the gradient vector at P. Verify that the maximum value of g equals the magnitude of the gradient vector at P.

Solution to (a): We see that

$$\begin{aligned} f_x(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial x}(1+4x^2+3y^2) = \tfrac{8x}{1+4x^2+3y^2} \\ f_y(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial y}(1+4x^2+3y^2) = \tfrac{6y}{1+4x^2+3y^2} \end{aligned} \tag{4}$$

$$\to \nabla f(x,y) = \langle \frac{8x}{1+4x^2+3y^2}, \frac{6y}{1+4x^2+3y^2} \rangle. \tag{5}$$

$$\nabla f(\frac{3}{4}, -\sqrt{3}) = \langle \frac{6}{1+\frac{9}{4}+9}, \frac{-6\sqrt{3}}{1+\frac{9}{4}+9} \rangle = \boxed{\langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle}. \tag{6}$$

Solution to (b): We recall that $\nabla f(P)$ points in the direction of maximum increase from P. Since $\nabla f(P)$ is in the fourth quadrant, we see that

$$\theta_{\max} = \tan^{-1}(\frac{\frac{-24\sqrt{3}}{49}}{\frac{24}{49}}) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}. \tag{7}$$

is the angle associated with the direction of maximum increase. Since $-\nabla f(P)$ points in the direction of maximum decrease from P, we see that $\theta_{\min} = \theta_{\max} + \pi = \frac{2\pi}{3}$ is the angle associated with the direction of maximum decrease. Since the directions of no change are orthogonal to $\nabla f(P)$ (and to $-\nabla f(P)$), we see that $\theta_1 = \theta_{\max} + \frac{\pi}{2} = \frac{5\pi}{6}$ and $\theta_2 = \theta_{\max} - \frac{\pi}{2} = -\frac{\pi}{6}$ are the angles associated to the directions of zero change.

Solution to (c): We recall that $\vec{u}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle$ is the unit vector associated with the angle θ. We also recall that for any unit vector $\vec{u}$, we have that

$$d_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}, \text{ so} \tag{8}$$

$$g(\theta) = d_{\vec{u}(\theta)} f(P) = \nabla f(P) \cdot \vec{u}(\theta) = \langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle \cdot \langle \cos(\theta), \sin(\theta) \rangle \tag{9}$$

$$= \boxed{\frac{24}{49}\cos(\theta) - \frac{24\sqrt{3}}{49}\sin(\theta)}. \tag{10}$$

Solution to (d): We see that

$$g'(\theta) = -\frac{24}{49}\sin(\theta) - \frac{24\sqrt{3}}{49}\cos(\theta) \rightarrow \tag{11}$$

$$g'(\theta) = 0 \Leftrightarrow -\frac{24}{49}\sin(\theta) = \frac{24\sqrt{3}}{49}\cos(\theta) \Leftrightarrow \tan(\theta) = -\sqrt{3} \Leftrightarrow \tag{12}$$

$$\theta = -\frac{\pi}{3}, \frac{2\pi}{3} \tag{13}$$

We see that

$$g''(\theta) = -\frac{24}{49}\cos(\theta) + \frac{24\sqrt{3}}{49}\sin(\theta) \tag{14}$$

$$\rightarrow g''(-\frac{\pi}{3}) = -\frac{24}{49}\cos(-\frac{\pi}{3}) + \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = -\frac{48}{89} < 0. \tag{15}$$

The second derivative test shows us that $g(\theta)$ has a local maximum at $\theta = -\frac{\pi}{3}$.

$$g(-\frac{\pi}{3}) = \frac{24}{49}\cos(-\frac{\pi}{3}) - \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = \frac{48}{49}. \tag{16}$$

we see that g attains its maximum value of $\frac{48}{49}$ on $[0, 2\pi]$ at $\theta = -\frac{\pi}{3}$.

Solution to (e): From parts b and d we have already seen that the value of θ that maximizes $g(\theta)$ is the same as the angle θ associated with the direction of maximum increase. To finish, we just note that

$$|\nabla f(\frac{3}{4}, -\sqrt{3})| = |\langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle| = \frac{24}{49}|\langle 1, -\sqrt{3} \rangle| \tag{17}$$

$$= \frac{24}{49}\sqrt{1^2 + (-\sqrt{3})^2} = \frac{48}{49}. \tag{18}$$

Problem 4: Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function

$$\varphi(x, y) = \sqrt{x^2 + y^2}, \quad \text{for } x^2 + y^2 \leq 9, (x, y) \neq (0, 0). \tag{19}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

Solution: Firstly, we see that

$$\vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \rangle. \tag{20}$$

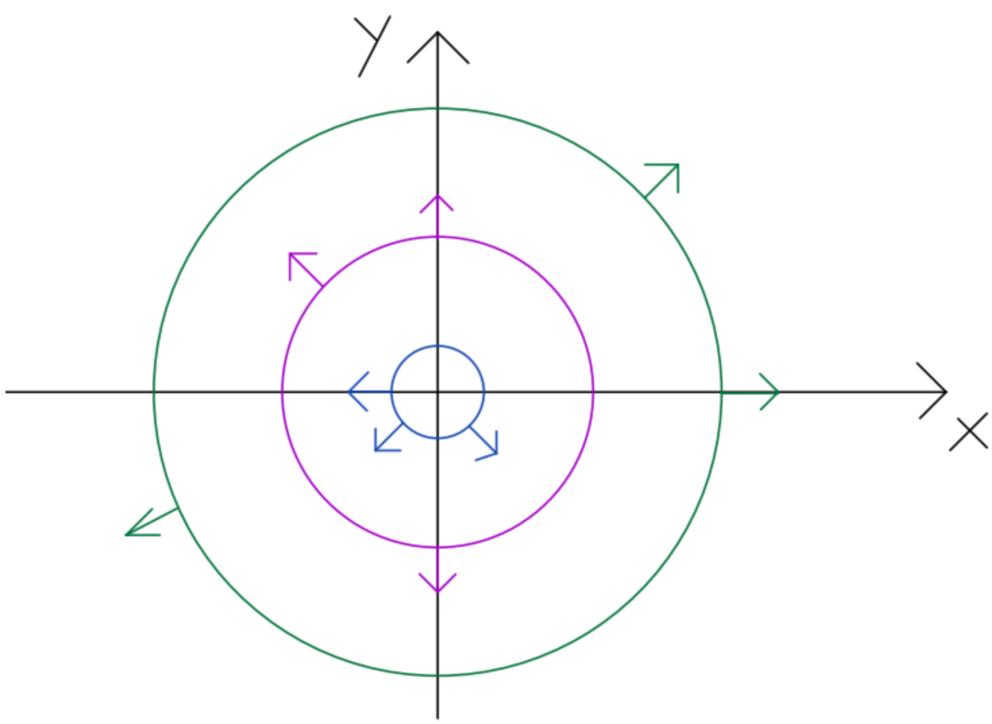
Next, we recall that the level curves of ϕ are the curves of the form $\phi(x, y) = c$ for some constant c. We see that

$$\phi(x, y) = c \Leftrightarrow \sqrt{x^2 + y^2} = c \Leftrightarrow x^2 + y^2 = c^2, \tag{21}$$

so the level curves of ϕ are circles centered at the origin. We recall that at a given point (x, y) the vector $\nabla\varphi(x, y)$ is perpendicular to the level curve that passes through (x, y), and we also observe that for any (x, y) we have

$$|\nabla\varphi(x, y)| = \sqrt{(\frac{x}{\sqrt{x^2 + y^2}})^2 + (\frac{y}{\sqrt{x^2 + y^2}})^2} = 1, \tag{22}$$

so we obtain the sketch below of some vectors from the gradient field and some level curves.



Problem 5: The electric field due to a point charge of strength Q at the origin has a potential function $V(x,y,z) = kQ/r$, where $r^2 = x^2+y^2+z^2$ is the square of the distance between a variable point $P(x,y,z)$ at the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E}(x,y,z) = -\nabla V(x,y,z)$.

(a) Show that

$$\mathbf{E}(x,y,z) = kQ\langle\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle. \tag{23}$$

(b) Show that $|\mathbf{E}| = kQ/r^2$. Explain why this relationship is called the inverse square law.

Solution to (a): We note that since r represents a distance, r is a nonnegative number, so

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \text{ (and not } -(x^2+y^2+z^2)^{\frac{1}{2}}). \tag{24}$$

It follows that

$$V(x,y,z) = kQ(x^2+y^2+z^2)^{-\frac{1}{2}} \rightarrow \tag{25}$$

$$\begin{aligned} V_x(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial x}(x^2+y^2+z^2) \\ &= -kQx(x^2+y^2+z^2)^{-\frac{3}{2}} \qquad = -kQxr^{-3} \\ V_y(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial y}(x^2+y^2+z^2) \\ &= -kQy(x^2+y^2+z^2)^{-\frac{3}{2}} \qquad = -kQyr^{-3} \\ V_z(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial z}(x^2+y^2+z^2) \\ &= -kQz(x^2+y^2+z^2)^{-\frac{3}{2}} \qquad = -kQzr^{-3} \end{aligned} \tag{26}$$

It is now clear that

$$\mathbf{E}(x,y,z) = -\nabla V(x,y,z) = -\langle V_x, V_y, V_z\rangle \tag{27}$$

$$= -\langle -kQxr^{-3}, -kQyr^{-3}, -kQzr^{-3}\rangle = kQ\langle\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle. \tag{28}$$

Solution to (b): We see that

$$|\mathbf{E}| = |kQ\langle\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle| = kQ|\langle\frac{x}{r^3},\frac{y}{r^3},\frac{z}{r^3}\rangle| = kQ\left((\frac{x}{r^3})^2+(\frac{y}{r^3})^2+(\frac{z}{r^3})^2\right)^{\frac{1}{2}} \tag{29}$$

$$= kQ\left(\frac{x^2+y^2+z^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{r^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{1}{r^4}\right)^{\frac{1}{2}} = \frac{kQ}{r^2}. \tag{30}$$

The fact that $|\mathbf{E}| = \frac{kQ}{r^2}$ is known as the inverse square law because the magnitude of the electric field $\mathbf{E}$ is proportional to the inverse of the square (or the square of the inverse) of the distance r.

Problem 6: Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all $(x, y, z) \in \mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

(a) We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = w(x, y) = f(x, y, z(x, y))$. Show that $\frac{\partial}{\partial x} w(x, y) = 18$.

(b) Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = w(x, z) = f(x, y(x, z), z)$. Show that $\frac{\partial}{\partial x} w(x, z) = 8$.

(c) Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

Solution to (a): Since $z = 4x - 2y$, we are lucky enough to see that $z(x, y) = 4x - 2y$ without even having to manipulate the original equation. We now see that

$$w = w(x, y) = f(x, y, z(x, y)) = 2x + 3y + 4(4x - 2y) = 18x - 5y \tag{31}$$

$$\Rightarrow \frac{\partial}{\partial x} w(x, y) = 18. \tag{32}$$

Solution to (b): Firstly, we observe that

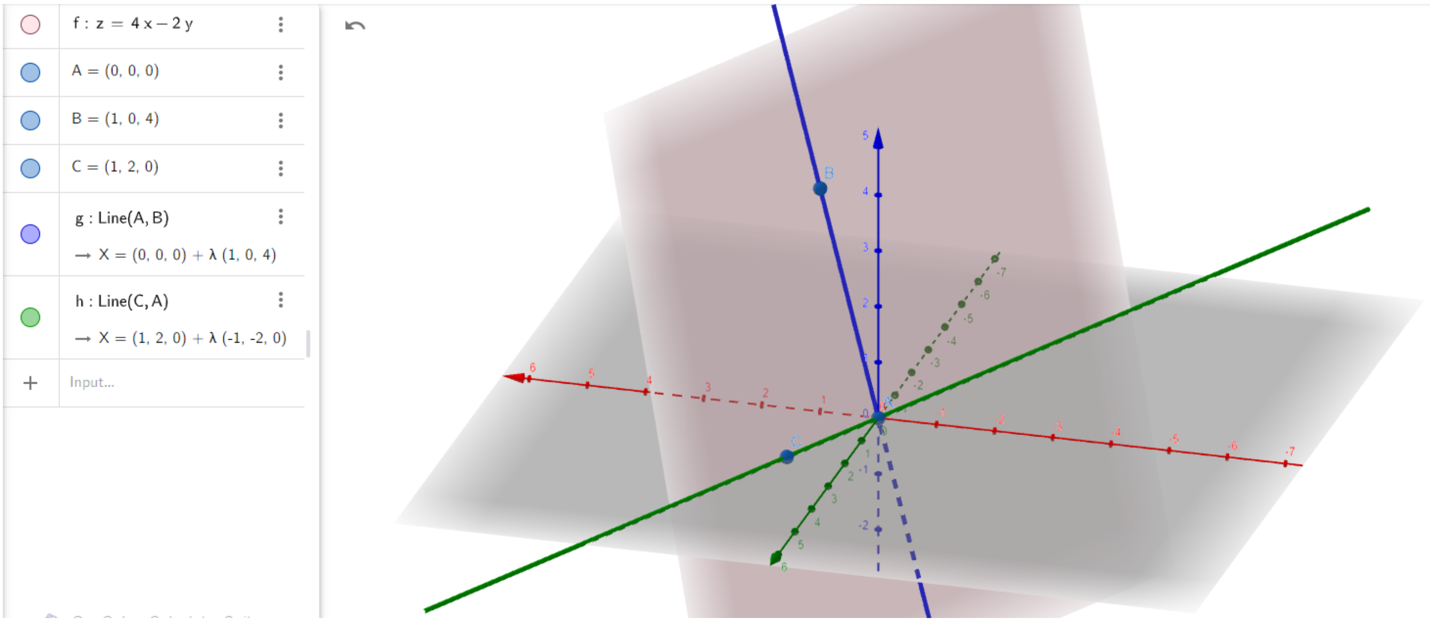
$$z = 4x - 2y \Rightarrow 2y = 4x - z \Rightarrow y = y(x, z) = 2x - \frac{1}{2}z. \tag{33}$$

We now see that

$$w = w(x, z) = f(x, y(x, z), z) = 2x + 3(2x - \frac{1}{2}z) + 4z = 8x + \frac{5}{2}z \tag{34}$$

$$\Rightarrow \frac{\partial}{\partial x} w(x, z) = 8. \tag{35}$$

Solution to (c): In our graph of $z = 4x - 2y$ we have also included graphs of the lines $(0 = 4x - 2y, z = 0)$ and $(z = 4x, y = 0)$, which are the lines residing within $z = 4x - 2y$ when you set $y = 0$ and $z = 0$ respectively. We do this to analyze what happens when calculating $\frac{\partial w}{\partial x}(0, 0, 0)$ (to have a concrete example) using the methods of parts (a) and (b).



We see that fixing $z = 0$ in part (a) to obtain $w(x, y)$ simply gives us the values of w over the line $(0 = 4x - 2y, z = 0)$. Similarly, fixing $y = 0$ in part (b) to obtain $w(x, y)$ simply gives us the values of w over the line $(z = 4x, y = 0)$. We now see that in part (a) we calculated the directional derivative of w in the direction of the line $(0 = 4x - 2y, z = 0)$ and in part (b) we calculated the directional derivative of w in the direction of the line $(z = 4x, y = 0)$. Said differently, part (a) showed us that $D_{\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle} w(x, y, z) = 18$ and part (b) showed us that $D_{\langle \frac{1}{\sqrt{17}}, 0, \frac{4}{\sqrt{17}}\rangle} w(x, y, z) = 8$.