Problem 1: Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.

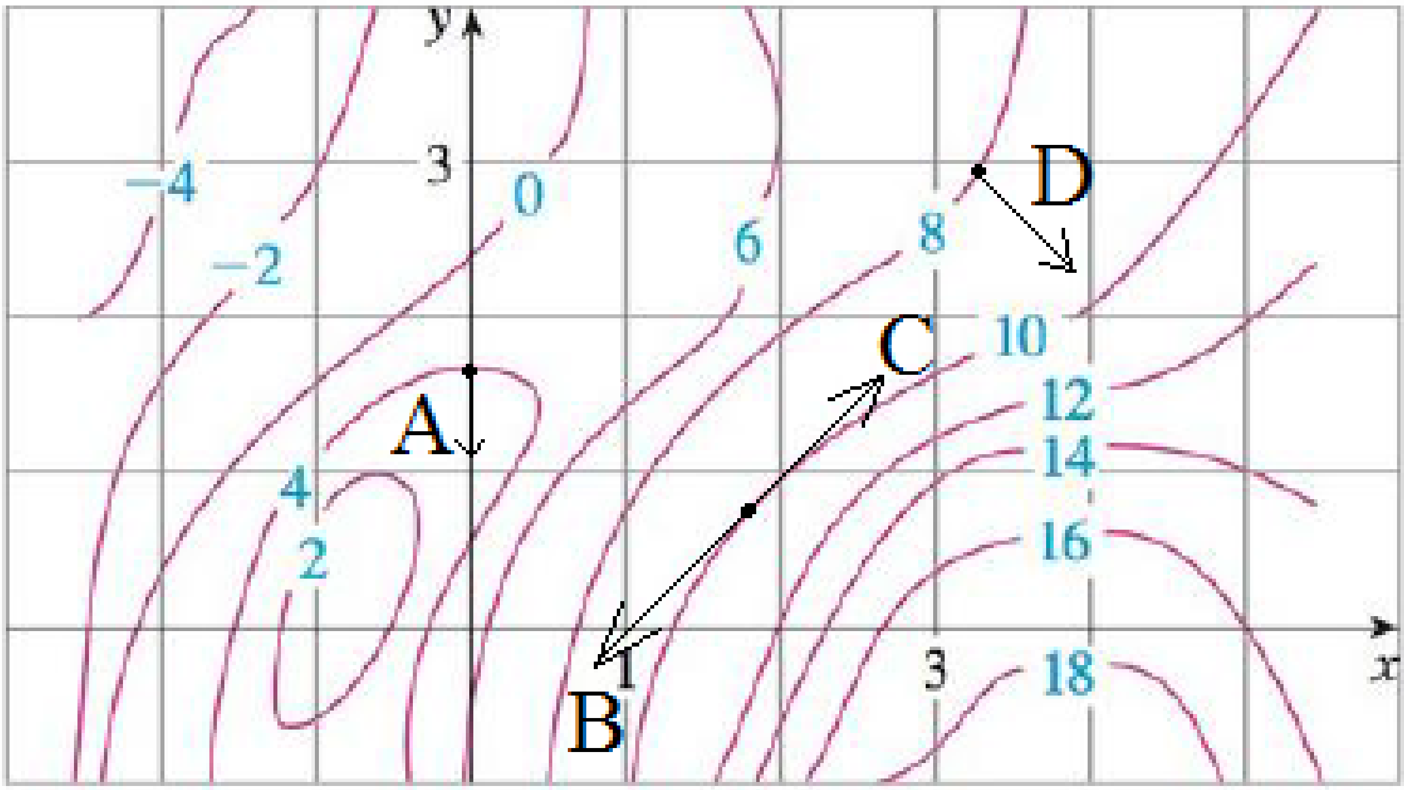

Figure 1: Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) (D) (E) None of the given vectors

Problem 2: Consider the function $f(x, y) = x^2 + y^2$ and and the point $P = (2, 3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2 \rangle$.

Problem 3: Consider the function $f(x, y) = \ln(1 + 4x^2 + 3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

(a) Find the gradient field $\nabla f(x, y)$ of $f(x, y)$ and then evaluate it at P.

(b) Find the angles θ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

(c) Write the directional derivative at P as a function of θ; call this function $g(\theta)$.

(d) Find the value of θ that maximizes $g(\theta)$ and find the maximum value.

(e) Verify that the value of θ that maximizes g corresponds to the direction of the gradient vector at P. Verify that the maximum value of g equals the magnitude of the gradient vector at P.

Problem 4: Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function

$$\varphi(x,y) = \sqrt{x^2+y^2}, \quad \text{for } x^2+y^2 \leq 9, (x,y) \neq (0,0). \tag{1}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

Problem 5: The electric field due to a point charge of strength Q at the origin has a potential function $V(x, y, z) = kQ/r$, where $r^2 = x^2 + y^2 + z^2$ is the square of the distance between a variable point $P(x, y, z)$ at the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E}(x, y, z) = -\nabla V(x, y, z)$.

(a) Show that

$$\mathbf{E}(x, y, z) = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle. \tag{2}$$

(b) Show that $|\mathbf{E}| = kQ/r^2$. Explain why this relationship is called the inverse square law.

Problem 6: Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all $(x, y, z) \in \mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

(a) We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = w(x, y) = f(x, y, z(x, y))$. Show that $\frac{\partial}{\partial x} w(x, y) = 18$.

(b) Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = w(x, z) = f(x, y(x, z), z)$. Show that $\frac{\partial}{\partial x} w(x, z) = 8$.

(c) Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.