Problem 1: Verify that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = 1. \tag{1}$$

Solution: We begin by reviewing one of the sum to product trigonometric identities. Observe that

$$\boxed{\sin(x)+\sin(y)} = \sin(\frac{x+y}{2}+\frac{x-y}{2}) + \sin(\frac{x+y}{2}-\frac{x-y}{2}) \tag{2}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(-\frac{x-y}{2}) + \sin(-\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{3}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(+\frac{x-y}{2}) - \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{4}$$

$$= \boxed{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}. \tag{5}$$

Recalling that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1, \tag{6}$$

we let $z = \frac{x+y}{2}$ and see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = \lim_{(x,y)\to(0,0)} \frac{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}{x+y} \tag{7}$$

$$\left(\lim_{(x,y)\to(0,0)} \frac{\sin(\frac{x+y}{2})}{\frac{x+y}{2}}\right)\left(\lim_{(x,y)\to(0,0)} \cos(\frac{x-y}{2})\right) \tag{8}$$

$$\left(\lim_{z\to 0}\frac{\sin(z)}{z}\right)\left(\lim_{(x,y)\to(0,0)} \cos(\frac{x-y}{2})\right) = (1)(\cos(\frac{0-0}{2})) = 1. \tag{9}$$

Problem 2: Consider the function

$$f(x,y) = \frac{xy^2}{x^2+y^4}. \tag{10}$$

(a) Show that if L is a line that passes through the origin, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = 0. \tag{11}$$

(b) Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \tag{12}$$

does not exist.

Solution to (a): Firstly, we see that if L is a line of the form $y = mx$ for some $m \in \mathbb{R}$, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4} \tag{13}$$

$$= \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0} \frac{m^2x}{1+m^4x^2} = 0. \tag{14}$$

The only line L left to consider is the line through the origin with infinite slope, which is just the line $x = 0$. In this case we see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0\cdot y^2}{0^2+y^4} = 0. \tag{15}$$

Solution to (b): In order to show that the limit in equation (12) does not exist we need to use the 2 path test. Based on part (a), we see that our second path needs cannot be a line. Thankfully, we only need to find a path P that results in any nonzero value when the limit is taken along P. If we try the parabolic path $y = x^2$, then we again get a value of 0 for the limit, but if we try the path $x = y^2$ then we get a value of $\frac{1}{2}$! In fact, we see that for $m \in \mathbb{R}$ and the path P_m given by $x = my^2$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f(x,y) = \lim_{y\to 0} f(my^2,y) = \lim_{y\to 0} \frac{(my^2)y^2}{(my^2)^2+y^4} \tag{16}$$

$$= \lim_{y\to 0} \frac{my^4}{m^2y^4+y^4} = \lim_{y\to 0} \frac{m}{m^2+1}. \tag{17}$$

Since the range of the function $g(m) = \frac{m}{m^2+1}$ is $[-1, 1]$, we see that the limit can take on any value between -1 and 1 if the correct path is chosen. While we only need 2 paths that result in different values to apply the 2 path test, it is amusing to see that we have found infinitely many paths that result in infinitely many different values.

Problem 3: Consider the function $f(x,y) = \sqrt{|xy|}$.

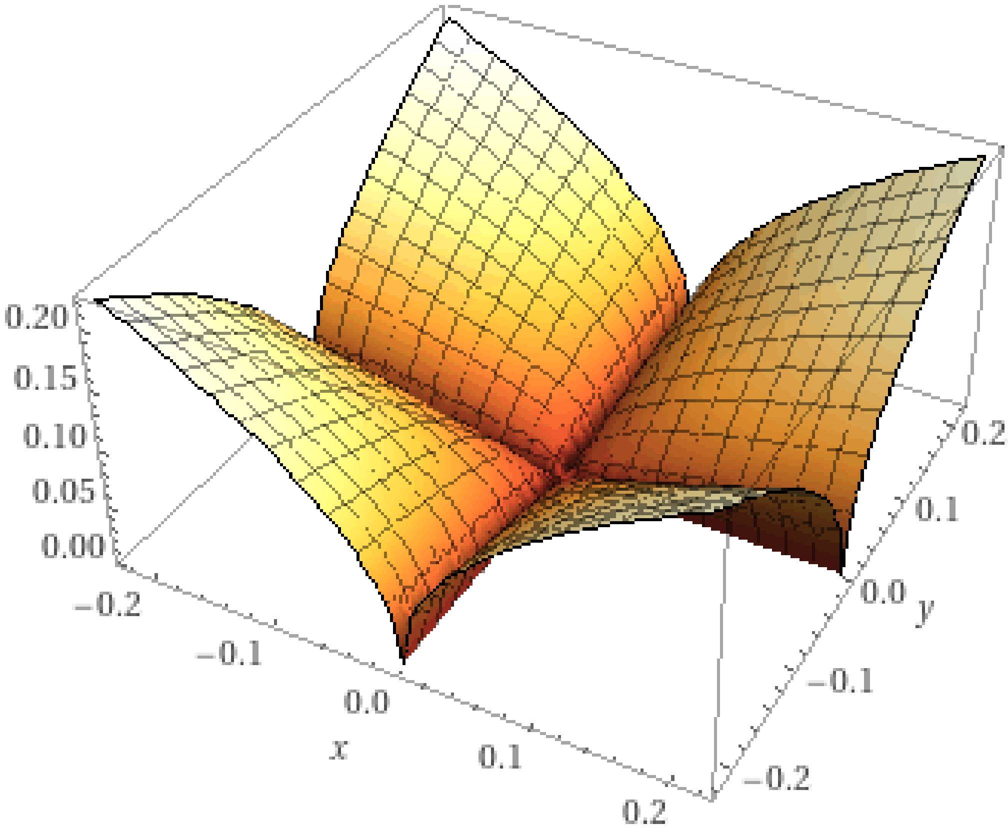

Figure 1: A graph of $z = \sqrt{|xy|}$.

(a) Is f continuous at $(0,0)$?

(b) Show that $f_x(0,0)$ and $f_y(0,0)$ exist by calculating their values.

(c) Determine whether f_x and f_y are continuous at $(0,0)$.

(d) Is f differentiable at $(0,0)$?

Solution to (a): $\boxed{\text{Yes.}}$ We will show that $f(x,y)$ is continuous on all of $\mathbb{R}^2$. The function $f_1(x,y) = xy$ is a continuous function since it is a polynomial function. The function $f_2(x) = |x|$ is also a continuous function, and the composition of continuous functions is again continuous, so we see that $f_3(x,y) := f_2(f_1(x,y)) = |xy|$ is a continuous function. Since $|xy|$ only takes on nonnegative values and the function $f_4(x) = \sqrt{x}$ is continuous on the domain $[0,\infty)$, we see that $f(x,y) = f_4(f_3(x,y))$ is indeed a continuous function.

Solution to (b): We see that

$$f_y(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0}\frac{\sqrt{|0\cdot y|}-\sqrt{|0\cdot 0|}}{y} = 0, \text{ and} \tag{18}$$

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{\sqrt{|x\cdot 0|}-\sqrt{|0\cdot 0|}}{x} = 0. \tag{19}$$

Solution to (c): We will show that neither of f_x and f_y are continuous at $(0,0)$. We note that for all $x,y>0$ we have $f(x,y)=\sqrt{|xy|}=\sqrt{xy}$. It follows that for $x,y>0$ we have

$$f_x(x,y) = \frac{\partial}{\partial x}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot y = \frac{1}{2}\sqrt{\frac{y}{x}},\text{ and} \tag{20}$$

$$f_y(x,y) = \frac{\partial}{\partial y}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}. \tag{21}$$

We now use the 2 path test to show that neither function is continuous. Let us consider the path P_m given by $y=mx$ with $m,x>0$ so that the path lies in the first quadrant. We see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_x(x,y) = \lim_{x\to 0^+} f_x(x,mx) = \lim_{x\to 0^+}\frac{1}{2}\sqrt{\frac{mx}{x}} = \frac{m}{2},\text{ and} \tag{22}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_y(x,y) = \lim_{x\to 0^+} f_y(x,mx) = \lim_{x\to 0^+}\frac{1}{2}\sqrt{\frac{x}{mx}} = \frac{1}{2m}. \tag{23}$$

We see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and 1 respectively for the value of $f_x(x,y)$ as (x,y) approaches $(0,0)$, so f_x is not continuous at $(0,0)$. Similarly, we see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and $\frac{1}{4}$ respectively for the value of $f_y(x,y)$ as (x,y) approaches $(0,0)$, so f_y is not continuous at $(0,0)$.

Solution to (d): $\boxed{\text{No.}}$ We begin by examining the directional derivative in the direction of the vector $\hat{u}=\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$ at $(0,0)$. We see that

$$D_{\hat{u}}f(0,0) = \lim_{t\to 0}\frac{f((0,0)+t\hat{u})-f(0,0)}{t} = \lim_{t\to 0}\frac{f(\frac{t}{\sqrt{2}},\frac{t}{\sqrt{2}})-0}{t} \tag{24}$$

$$= \lim_{t\to 0}\frac{\sqrt{|\frac{t^2}{2}|}}{t} = \lim_{t\to 0}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{25}$$

If f was differentiable at $(0,0)$, then we would have

$$D_{\hat{u}}f(0,0) = \nabla f(0,0)\cdot\hat{u} = \langle f_x(0,0),f_y(0,0)\rangle\cdot\langle\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle = 0. \tag{26}$$

Since this is not the case, we see that f is not differentiable at $(0,0)$.

Problem 4: Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x,t) = 2\sin(\pi x)\sin(\frac{\pi}{2}t)$, for $0 \leq x \leq 1$ and $t \geq 0$ (see figure 2). At a fixed point in time, the string forms a wave on $[0,1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

(a) What is the period of the motion in time?

(b) Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).

(c) At a fixed time, what point on the string is moving fastest?

(d) At a fixed position on the string, when is the strong moving fastest?

(e) Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).

(f) At a fixed time, where is the slope of the string greatest?

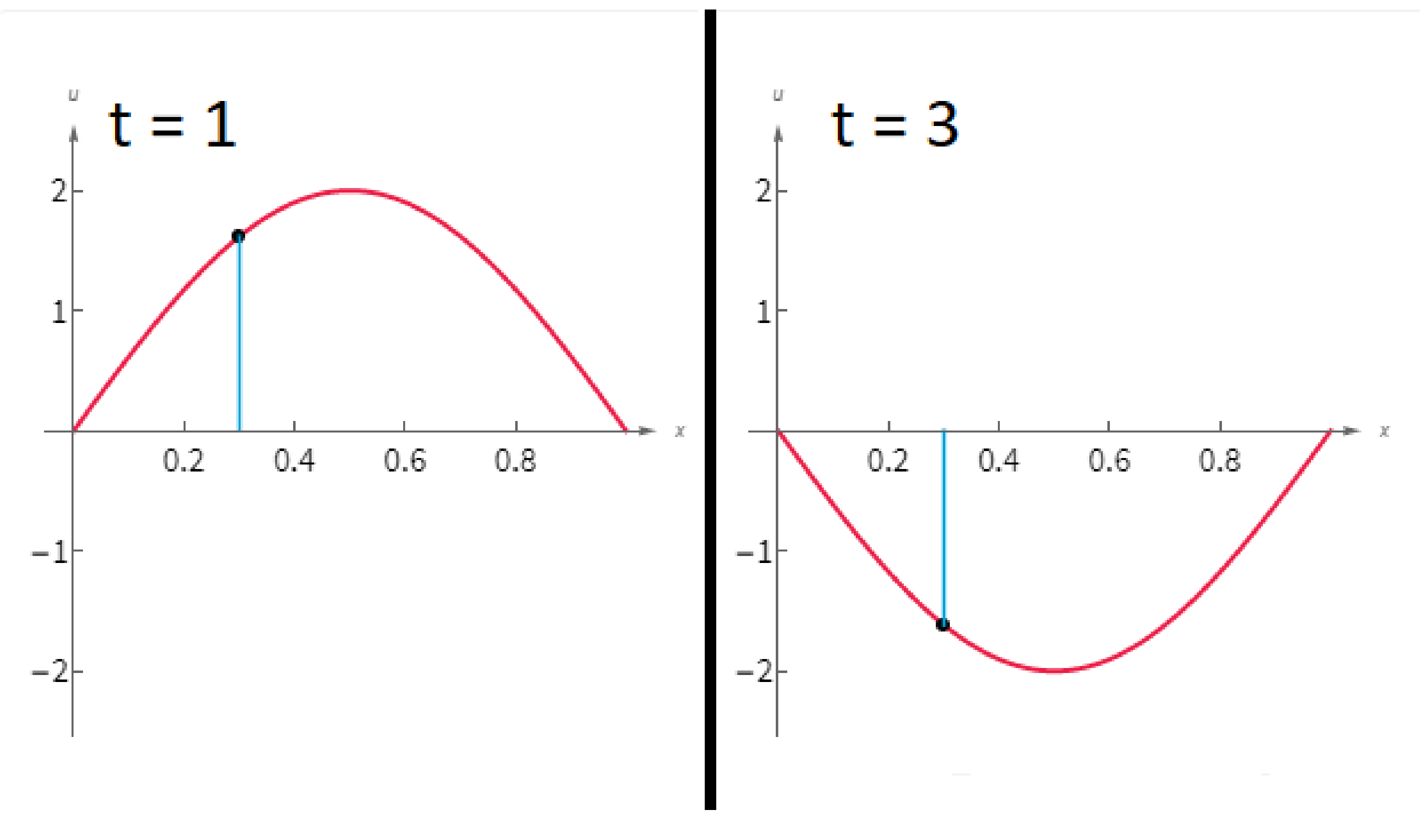

Figure 2: Snapshots of the wave at times $t = 1$ and $t = 3$.

Solution to (a): We begin by recalling that the period of sin (as well as cos, tan, csc, sec, and cot) is 2π, i.e., $\sin(y) = \sin(y + 2\pi)$ for all $y \in \mathbb{R}$. We now want to find the smallest $p > 0$ such that

$$\sin(\frac{\pi}{2}(t+p)) = \sin(\frac{\pi}{2}t), \tag{27}$$

which will happen if

$$\frac{\pi}{2}t + \frac{\pi}{2}p = \frac{\pi}{2}t + 2\pi \Rightarrow \boxed{p = 4}. \tag{28}$$

Solution to (b): If we fix a position of $x = x_0$, then $v(x_0, t)$, the rate of change of displacement with respect to time is given by the t partial derivative of f. We now observe that

$$v(x_0, t) = f_t(x_0, t) = \frac{\partial}{\partial t} 2\sin(\pi x_0)\sin(\frac{\pi}{2}t) \overset{*}{=} 2\sin(\pi x_0)\frac{\partial}{\partial t}\sin(\frac{\pi}{2}t) \tag{29}$$

$$= 2\sin(\pi x_0)\Big(\frac{\pi}{2}\cos(\frac{\pi}{2}t)\Big) = \boxed{\pi\sin(\pi x_0)\cos(\frac{\pi}{2}t)}, \tag{30}$$

where equation * follows from the fact that $2\sin(\pi x_0)$ is a constant.

Solution to (c): Since speed is just the absolute value of velocity, it suffices to optimize the velocity function $v(x, t)$. In the end, the largest possible speed is going to either be the largest possible velocity, or the absolute value of the smallest possible velocity. Since we are fixing a time $t = t_0$, we seek to optimize the function $h(x) := v(x, t_0)$ with respect to x, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $h(x)$ in the interval $[0, 1]$. We now see that

$$0 = \frac{d}{dx}h(x) = \frac{\partial}{\partial x}v(x, t_0) = \frac{\partial}{\partial x}\pi\sin(\pi x)\cos(\frac{\pi}{2}t_0) \overset{*}{=} \pi\cos(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\sin(\pi x) \tag{31}$$

$$= \pi\cos(\frac{\pi}{2}t_0)\Big(\pi\cos(\pi x)\Big) = \pi^2\cos(\frac{\pi}{2}t_0)\cos(\pi x) \Rightarrow 0 = \cos(\frac{\pi}{2}t_0)\cos(\pi x), \tag{32}$$

where equation * follows from the fact that $\pi\cos(\frac{\pi}{2}t_0)$ is a constant. We now observe that if $\cos(\frac{\pi}{2}t_0) = 0$, then $v(x, t_0) = 0$ for all $x \in [0, 1]$, so in this situation every point on the string is the fastest moving point since every point is moving (or not moving since the velocity is 0) at the same speed. Having fully resolved the situation when $\cos(\frac{\pi}{2}t_0) = 0$, we proceed to the remaining situation in which $\cos(\frac{\pi}{2}t_0) \neq 0$, in which case we are allowed to divide both sides of the right hand equation in (32) by $\cos(\frac{\pi}{2}t_0)$ to see that $0 = \cos(\pi x)$. Recalling that $\cos(y) = 0$ if and only if $y = \frac{\pi}{2} + n\pi$ for some integer n, we see that $x \in \{\cdots - \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$. Recalling that $x \in [0, 1]$ we see that $x = \frac{1}{2}$ is the only critical point. Since the end points of the domain of x are 0 and 1, we observe that $v(0, t_0) = v(1, t_0) = 0$ (which should not be a surprise since the end points of our string are not moving) and that $v(\frac{1}{2}, t_0) = 2\cos(\frac{\pi}{2}t_0)$. Lastly, to put together the results of our preceding two cases we recall that $\cos(\frac{\pi}{2}t_0) = 0$ if and only if $\frac{\pi}{2}t_0 = \frac{\pi}{2} + n\pi$ for some integer n, which happens if and only if $t = 2n+1$. Since $t \geq 0$, we see that this happens if and only if t is a positive odd integer. In conclusion, the fastest moving point on the string at time $t = t_0$ is

$$\boxed{\begin{cases} x = \frac{1}{2} & \text{if } t \text{ is not an odd integer} \\ \text{all } x \in [0,1] & \text{else} \end{cases}}. \tag{33}$$

Solution to (d): As in part (c) we begin by optimizing the velocity function $v(x,t)$. Since we are fixing a position $x = x_0$, we seek to optimize the function $q(t) := v(x_0,t)$ with respect to t, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $q(t)$ over $[0,\infty)$. We now see that

$$0 = \frac{d}{dt}q(t) = \frac{\partial}{\partial t}v(x_0,t) = \frac{\partial}{\partial t}\pi\sin(\pi x_0)\cos(\frac{\pi}{2}t) \overset{*}{=} \pi\sin(\pi x_0)\frac{\partial}{\partial t}\cos(\frac{\pi}{2}t) \tag{34}$$

$$= \pi\sin(\pi x_0)\Big(-\frac{\pi}{2}\sin(\frac{\pi}{2}t)\Big) = -\frac{\pi^2}{2}\sin(\pi x_0)\sin(\frac{\pi}{2}t) \Rightarrow 0 = \sin(\pi x_0)\sin(\frac{\pi}{2}t), \tag{35}$$

where equation $*$ follows from the fact that $\pi\sin(\pi x_0)$ is a constant. We now observe that if $\sin(\pi x_0) = 0$ for some $x_0 \in [0,1]$, then $x_0 = 0,1$. Noting that $v(0,t) = v(1,t) = 0$ for all $t \geq 0$ (which makes sense since the endpoints of the string are fixed) we see that any time $t \geq 0$ results in the fastest velocity if $x_0 = 0,1$. We now proceed to the situation in which $x_0 \in (0,1)$, so $\sin(\pi x_0) \neq 0$ and we can divide the right hand equation in (35) by $\sin(\pi x_0)$ to see that $0 = \sin(\frac{\pi}{2}t)$. Recalling that $\sin(y) = 0$ if and only if $y = n\pi$ for some integer n, we see that $t \in \{\cdots -4,-2,0,2,4,\cdots\}$. Recalling that $t \geq 0$ we see that $\{0,2,4,\cdots\}$ are all of the critical points, and this set of critical points coincidentally (luckily) includes the endpoint 0 of our region. Observing that $v(x_0,2n) = \pi\sin(\pi x_0)\cos(n\pi) = (-1)^n\pi\sin(\pi x_0)$, so the largest speed is $\pi|\sin(\pi x_0)|$. In conclusion, the fastest speed of the point $x = x_0$ on the string is attained at

$$\boxed{\begin{cases} \text{Every } t \geq 0 & \text{if } x_0 = 0,1 \\ \{0,2,4,\cdots\} & \text{else} \end{cases}}. \tag{36}$$

Interestingly, we note that if $t \in \{0,2,4,\cdots\}$ then the string is back at equilibrium (every point has 0 displacement) and if $t \in \{1,3,5,\cdots\}$ then the string is in an extreme state in which every point has its maximum possible displacement.

Solution to (e): If we fix a time $t = t_0$, then $s(x,t_0)$, the rate of change of the displacement with respect to position (slope of the string) at a constant time is given by the x partial derivative of f. We now observe that

$$s(x,t_0) = \frac{\partial}{\partial x}f(x,t_0) = \frac{\partial}{\partial x}2\sin(\pi x)\sin(\frac{\pi}{2}t_0) \overset{*}{=} 2\sin(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\sin(\pi x) \tag{37}$$

$$= 2\sin(\frac{\pi}{2}t_0)\Big(\pi\cos(\pi x)\Big) = \boxed{2\pi\cos(\pi x)\sin(\frac{\pi}{2}t_0)}. \tag{38}$$

Solution to (f): As in parts (c) and (d) we begin by optimizing the slope function $s(x,t)$. Since we are fixing a time $t = t_0$, we seek to optimize the function $r(x) := s(x,t_0)$ with respect to x, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $r(x)$ over $[0,1]$. We now see that

$$0 = \frac{d}{dx}r(x) = \frac{\partial}{\partial x}s(x,t_0) = \frac{\partial}{\partial x}2\pi\cos(\pi x)\sin(\frac{\pi}{2}t_0) \overset{*}{=} 2\pi\sin(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\cos(\pi x) \tag{39}$$

$$= 2\pi\sin(\frac{\pi}{2}t_0)\Big(-\pi\sin(\pi x)\Big) = -2\pi\sin(\pi x)\sin(\frac{\pi}{2}t_0) \Rightarrow 0 = \sin(\pi x)\sin(\frac{\pi}{2}t_0), \tag{40}$$

where equation * follows from the fact that $2\pi\sin(\frac{\pi}{2}t_0)$ is a constant. As we saw in part (d), $\sin(\frac{\pi}{2}t_0) = 0$ for $t_0 \geq 0$ if and only if $t_0 \in \{0,2,4,\cdots\}$, and in this situation we see that $s(x,t_0) = 0$ for all x, so every x attains the greatest slope. We now consider the situation in which $t_0 \notin \{0,2,4,\cdots\}$ and divide the right hand equation of 40 by $\sin(\frac{\pi}{2}t_0)$ to see that $0 = \sin(\pi x)$. As before, we deduce that $x \in [0,1]$ must be an integer, so $x = 0,1$. Observing that

$$s(0,t_0) = 2\pi\cos(0)\sin(\frac{\pi}{2}t_0) = 2\pi\sin(\frac{\pi}{2}t_0) \text{ and} \tag{41}$$

$$s(1,t_0) = 2\pi\cos(\pi)\sin(\frac{\pi}{2}t_0) = -2\pi\sin(\frac{\pi}{2}t_0), \tag{42}$$

we see that the largest slope in this case is $2\pi|\sin(\frac{\pi}{2}t_0)|$, which occurs at 0 if $\sin(\frac{\pi}{2}t_0) > 0$ and at 1 if $\sin(\frac{\pi}{2}t_0) < 0$. In conclusion, the greatest slope of a point on the string at time $t = t_0$ is attained at

$$\boxed{\begin{cases} \text{Every } x \in [0,1] & \text{if } t_0 \in \{0,2,4,\cdots\} \\ x = 0 & \text{if } t_0 \in (2n, 2n+1) \text{ for some integer } n \\ x = 1 & \text{if } t_0 \in (2n+1, 2n+2) \text{ for some integer } n \end{cases}}. \tag{43}$$