Problem 1: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) The position, unit tangent, and principal unit normal vectors ($\vec{r}, \hat{T}$, and $\hat{N}$) at a point lie in the same plane.

(b) The vectors $\hat{T}$ and $\hat{N}$ at a point depend on the orientation of a curve.

(c) The curvature at a point depends on the orientation of a curve.

(d) An object with unit speed ($|\vec{v}| = 1$) on a circle of radius R has an acceleration of $\vec{a} = \frac{1}{R}\hat{N}$.

(e) If the speedometer of a car reads a constant 60 mi/hr, the car is not accelerating.

(f) A curve in the xy-plane that is concave up at all points has positive torsion.

(g) A curve with large curvature also has large torsion.

Solution to (a): False. To see a concrete counterexample, we simply consider $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle$. Since $\hat{T}(t) = \vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$, we see that $\hat{N} = \langle -\cos(t), -\sin(t), 0\rangle$. Since $\hat{T}(t)$ and $\hat{N}(t)$ are always in the xy-plane, but $\vec{r}(t)$ is never in the xy-plane, we see that we have indeed produced a counterexample.

In fact, we can generalize the idea behind the previous counterexample. If $\vec{r_2}(t) = \vec{r_1}(t) + \vec{v_0}$ for some constant vector $\vec{v_0}$, then $\hat{T_1} = \hat{T_2}$ and $\hat{N_1} = \hat{N_2}$, so if $\vec{r_1}(t_0), \hat{T_1}(t_0), \hat{N_1}(t_0)$ are coplanar for some t_0, then we can take $\vec{v_0}$ to be any vector that is outside of the plane P of $\hat{T_1}(t_0)$ and $\hat{N_1}(t_0)$ so that $\vec{r_2}(t_0) = \vec{r_1}(t_0) + \vec{v_0}$ will be outside of P. Since P is still the plane of $\hat{T_2}(t_0)$ and $\hat{N_2}(t_0)$, we see that $\vec{r_2}(t_0), \hat{T_2}(t_0)$, and $\hat{N_2}(t_0)$ are not coplanar. What we have essentially done is use the fact that $\hat{T}$ and $\hat{N}$ don't change as a result of a translation, so we can use a translation to ensure that $\vec{r}(t_0), \hat{T}(t_0)$, and $\hat{N}(t_0)$ are not coplanar at a given point t_0 regardless of the initial curve $\vec{r}(t)$.

Solution to (b): True. Consider for example the curves $y = x^2$ and $x = y^2$. The unit tangent vector to $y = x^2$ at the point $(0, 0)$ is either $(1, 0)$ or $(-1, 0)$ depending on the parameterization that is used. Similarly, the unit tangent vector to $x = y^2$ at $(0, 0)$ is either $(0, 1)$ or $(0, -1)$ depending on the parameterization that is used. We see that the cure $x = y^2$ is the curve $y = x^2$ rotated 90° clockwise, and that this will result in the unit tangent vector at $(0, 0)$ being rotated by 90° as shown in Figure 1. Similar considerations show that the orientation of a curve also affects $\hat{N}$.

Solution to (c): False. The curvature measures the rate at which the unit tangent vector $\hat{T}$ changes direction. If you alter a curves orientation via translations and rotations, then the translations do not affect the unit tangent vectors, but the rotations will rotate all of the unit tangent vectors as well. Since the same rotation is being applied to all of the unit tangent vectors, the rate at which their directions change will remain the same. For example, we can see in Figure 1 that the curvature of $y = x^2$ at the point (x_0, y_0) is the same as the curvature of $x = y^2$ at the point $(y_0, -x_0)$.

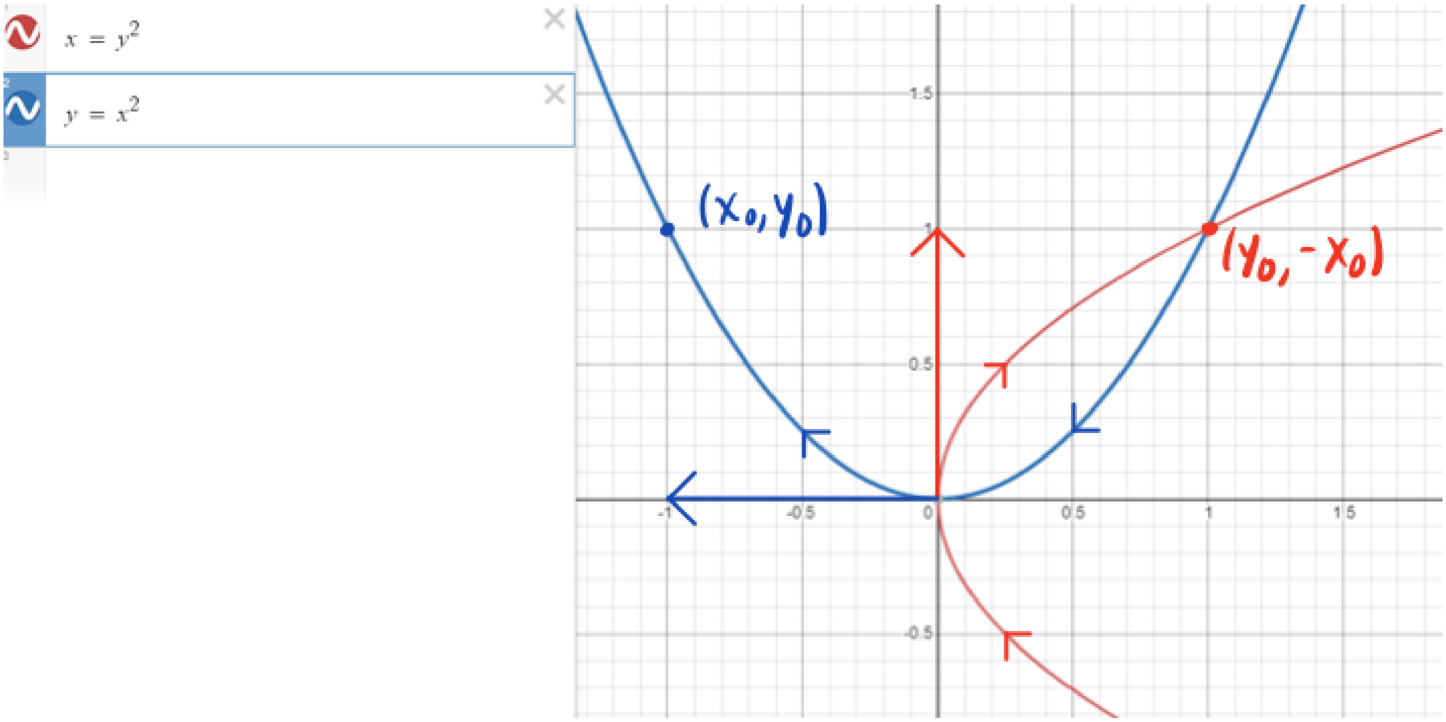


Figure 1: A comparison of $y = x^2$ and $x = y^2$.

Solution to (d): True. We present two verifications of this claim. For the first verification, we use the tangential and normal components of acceleration. In particular, we recall that

$$\vec{a}(t) = \frac{d^2s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2\hat{N}. \tag{1}$$

Since $\frac{ds}{dt} = |\vec{v}(t)| = 1$, we see that $\frac{d^2s}{dt^2} = 0$. Since a circle of radius R has curvature $\kappa = \frac{1}{R}$, we see that $\vec{a}(t) = \kappa\hat{N} = \frac{1}{R}\hat{N}(t)$.

Our next verification will be a direct calculation of $\vec{a}(t)$. Since the acceleration does not depend on where the circle is cetered, we may assume that we are working with a circle centered at $(0,0)$ in the xy-plane. Since $|\vec{v}(t)| = 1$, we are working with a paramaterization by arclength of our circle of radius R, which is given by

$$\vec{r}(t) = \langle R\cos(\frac{t}{R}), R\sin(\frac{t}{R})\rangle, 0 \leq t \leq 2\pi R. \tag{2}$$

We see that

$$\hat{T}(t)=\vec{v}(t)=\langle -\sin(\frac{t}{R}),\cos(\frac{t}{R})\rangle \text{ and } \vec{a}(t)=\langle -\frac{1}{R}\cos(\frac{t}{R}),-\frac{1}{R}\sin(\frac{t}{R})\rangle. \quad (3)$$

Since $|\hat{T}'(t)|=|\vec{a}(t)|=\frac{1}{R}$, we see that

$$\hat{N}(t)=\frac{\hat{T}'(t)}{|\hat{T}'(t)|}=\langle -\cos(\frac{t}{R}),-\sin(\frac{t}{R})\rangle, \quad (4)$$

so we do indeed have that $\vec{a}(t)=\frac{1}{R}\hat{N}(t)$.

Solution to (e): False. While the speed of the car is not changing, the car could still be changing its direction of motion, which would mean that the car is accelerating. This is easily seen to be the case if the cars trajectory is modeled by $\vec{r}(t)=\langle 60\cos(t),60\sin(t)\rangle$ since $\vec{v}(t)=\langle -60\sin(t),60\cos(t)\rangle$ satisfies $|\vec{v}(t)|=60$ and $\vec{a}(t)=\langle -60\cos(t),-60\sin(t)\rangle\neq\vec{0}$.

Solution to (f): False. The torsion at a point $\vec{r}(t_0)$ measures the rate at which the curve $\vec{r}(t)$ twists out of the plane determined by $\hat{T}(t_0)$ and $\hat{N}(t_0)$. If the curve $\vec{r}(t)$ is contained in the xy-plane (regardless of whether it is concave up or even convex) then $\hat{T}(t_0)$ and $\hat{N}(t_0)$ will be contained in the xy-plane as well for every t_0, so the torision will always be 0 since the curve does not twist out of the xy-plane at all.

Solution to (g): False. We recall that a circle in the xy-plane (or any other plane) of radius r has a curvature of $\kappa=\frac{1}{r}$. We already saw in part (f) that any such circle has 0 torsion at all points, regardless of the radius r. As r gets closer to 0, κ grows larger without bound, but the torsion is always 0.

Problem 2: Compute the unit binormal vector $\hat{B}$ and torsion τ of the curve parameterized by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), -t\rangle, t \in \mathbb{R}(-\infty < t < \infty)$.

Solution: Since $\hat{B} = \hat{T} \times \hat{N}$ and $\tau = -\frac{d\hat{B}}{ds} \cdot \hat{N} = -\frac{1}{|\vec{v}(t)|}\frac{d\hat{B}}{dt} \cdot \hat{N}$, we see that we should begin by calculating $\hat{T}(t)$ and $\hat{N}(t)$. To this end, we see that

$$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), -1\rangle \tag{5}$$

$$\to |\vec{r}\,'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + (-1)^2} = \sqrt{5}. \tag{6}$$

$$\to \hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\frac{2}{\sqrt{5}}\sin(t), \frac{2}{\sqrt{5}}\cos(t), -\frac{1}{\sqrt{5}}\rangle. \tag{7}$$

Recalling that $\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$, we see that

$$\hat{T}'(t) = \langle -\frac{2}{\sqrt{5}}\cos(t), -\frac{2}{\sqrt{5}}\sin(t), 0\rangle \tag{8}$$

$$\to |\hat{T}'(t)| = \sqrt{(-\frac{2}{\sqrt{5}}\cos(t))^2 + (-\frac{2}{\sqrt{5}}\sin(t))^2 + 0^2} = \frac{2}{\sqrt{5}} \tag{9}$$

$$\to \hat{N}(t) = \frac{\langle -\frac{2}{\sqrt{5}}\cos(t), -\frac{2}{\sqrt{5}}\sin(t), 0\rangle}{\frac{2}{\sqrt{5}}} = \langle -\cos(t), -\sin(t), 0\rangle. \tag{10}$$

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{2}{\sqrt{5}}\sin(t) & \frac{2}{\sqrt{5}}\cos(t) & -\frac{1}{\sqrt{5}} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} \tag{11}$$

$$= \hat{i}(\frac{2}{\sqrt{5}}\cos(t)\cdot 0 - (-\sin(t))\cdot(-\frac{1}{\sqrt{5}}))$$
$$-\hat{j}((-\frac{2}{\sqrt{5}}\sin(t))\cdot 0 - (-\cos(t))\cdot(-\frac{1}{\sqrt{5}}))$$
$$+\hat{k}((-\frac{2}{\sqrt{5}}\sin(t))\cdot(-\sin(t)) - (-\cos(t))\cdot\frac{2}{\sqrt{5}}\cos(t)) \quad (12)$$

$$= -\frac{1}{\sqrt{5}}\sin(t)\hat{i} + \frac{1}{\sqrt{5}}\cos(t)\hat{j} + \frac{2}{\sqrt{5}}(\sin^2(t)+\cos^2(t))\hat{k} \quad (13)$$

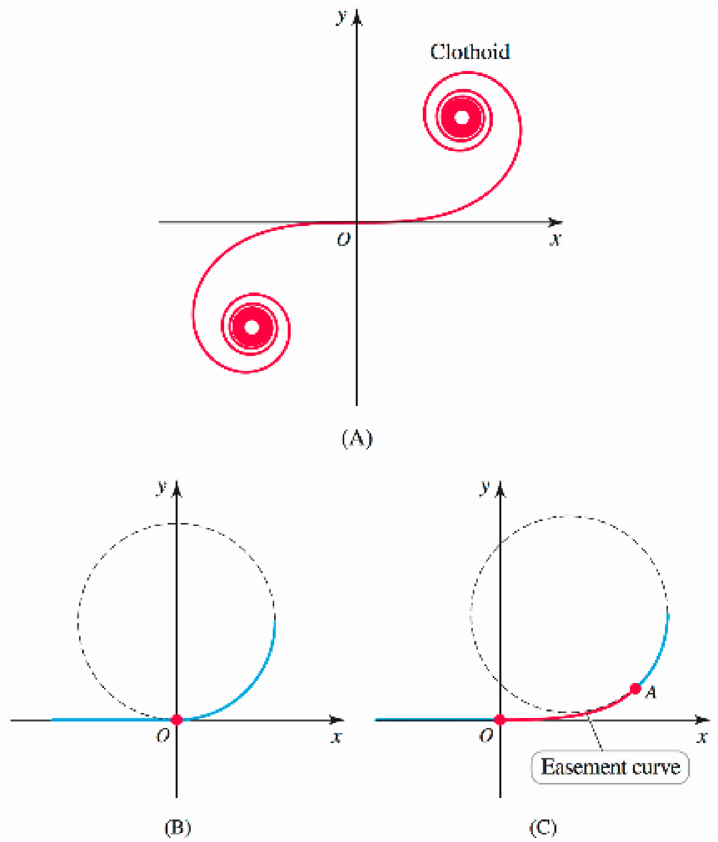
$$= \boxed{\langle -\frac{1}{\sqrt{5}}\sin(t), \frac{1}{\sqrt{5}}\cos(t), \frac{2}{\sqrt{5}}\rangle}. \quad (14)$$

Recalling that $|\vec{v}(t)| = |\vec{r}\,'(t)|$ we see that

$$\tau = \tau(t) = -\frac{1}{|\vec{v}(t)|}\frac{d\hat{B}}{dt}\cdot\hat{N} \quad (15)$$

$$= -\frac{1}{\sqrt{5}}\langle -\frac{1}{\sqrt{5}}\cos(t), -\frac{1}{\sqrt{5}}\sin(t), 0\rangle\cdot\langle -\cos(t), -\sin(t), 0\rangle = \boxed{-\frac{1}{5}} \quad (16)$$

Problem 3: The function $\vec{r}(t) = \langle \int_0^t \cos(\frac{1}{2}u^2)du, \int_0^t \sin(\frac{1}{2}u^2)du \rangle, t \in \mathbb{R}$ whose graph is called a **clothoid** or **Euler Spiral**, has applications in the design of railroad tracks, rollercoasters, and highways.



(a) A car moves from left to right on a straight highway, approaching a curve at the origin (Figure B). Sudden changes in curvature at the start of the curve may cause the driver to jerk the steering wheel. Suppose the curve starting at the origin is a segment of a circle of radius a. Explain why there is a sudden change in the curvature of the road at the origin.

(b) A better approach is to use a segment of a clothoid as an easement curve, in between the straight highway and a circle, to avoid sudden changes in curvature (Figure C). Assume the easement curve corresponds to the clothoid $\vec{r}(t)$, for $0 \leq t \leq 1.2$. Find the curvature of the easement curve as a function of t and explain why this curve eliminates the sudden change in curvature at the origin.

(c) Find the radius of a circle connected to the easement curve at point A (that corresponds to $t = 1.2$ on the curve $\vec{r}(t)$) so that the curvature of the circle matches the curvature of the easement curve at point A.

Solution to (a): We recall that straight lines have a curvature of 0 at every point and circles of radius a has a curvature of $\frac{1}{a}$ at every point.[1] Since $\frac{1}{a} \neq 0$ we see that there is a change in curvature when the line segment converts into a circular arc. Since the curvature κ is given by $\kappa = |\frac{d\hat{T}}{ds}|$, we recall that curvature tells us how quickly our path is changing direction. The driver on the curve in this part will notice that change in curvature as a jerk in their driving since the direction in which they are driving will change sharply instead of smoothly.

Solution to (b): Since $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{1}{|\vec{v}(t)|}\left|\frac{d\hat{T}}{dt}\right|$, we begin by calculating $\hat{T}(t)$. To this end, we see that

$$\vec{r}\,'(t) = \langle \frac{d}{dt}\int_0^t \cos(\frac{1}{2}u^2)du, \frac{d}{dt}\int_0^t \sin(\frac{1}{2}u^2)du\rangle = \langle \cos(\frac{1}{2}t^2), \sin(\frac{1}{2}t^2)\rangle. \tag{17}$$

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \sqrt{\cos^2(\frac{1}{2}t^2) + \sin^2(\frac{1}{2}t^2)} = 1. \tag{18}$$

Conveniently, we see that the parameterization for the clothoid that we were given happened to be a parameterization by arclength, so we may interchange s and t in this situation. We now see that

$$\kappa(t) = \kappa(s) = \left|\frac{\hat{T}(s)}{ds}\right| = |\langle -t\sin(\frac{1}{2}t^2), t\cos(\frac{1}{2}t^2)\rangle| \tag{19}$$

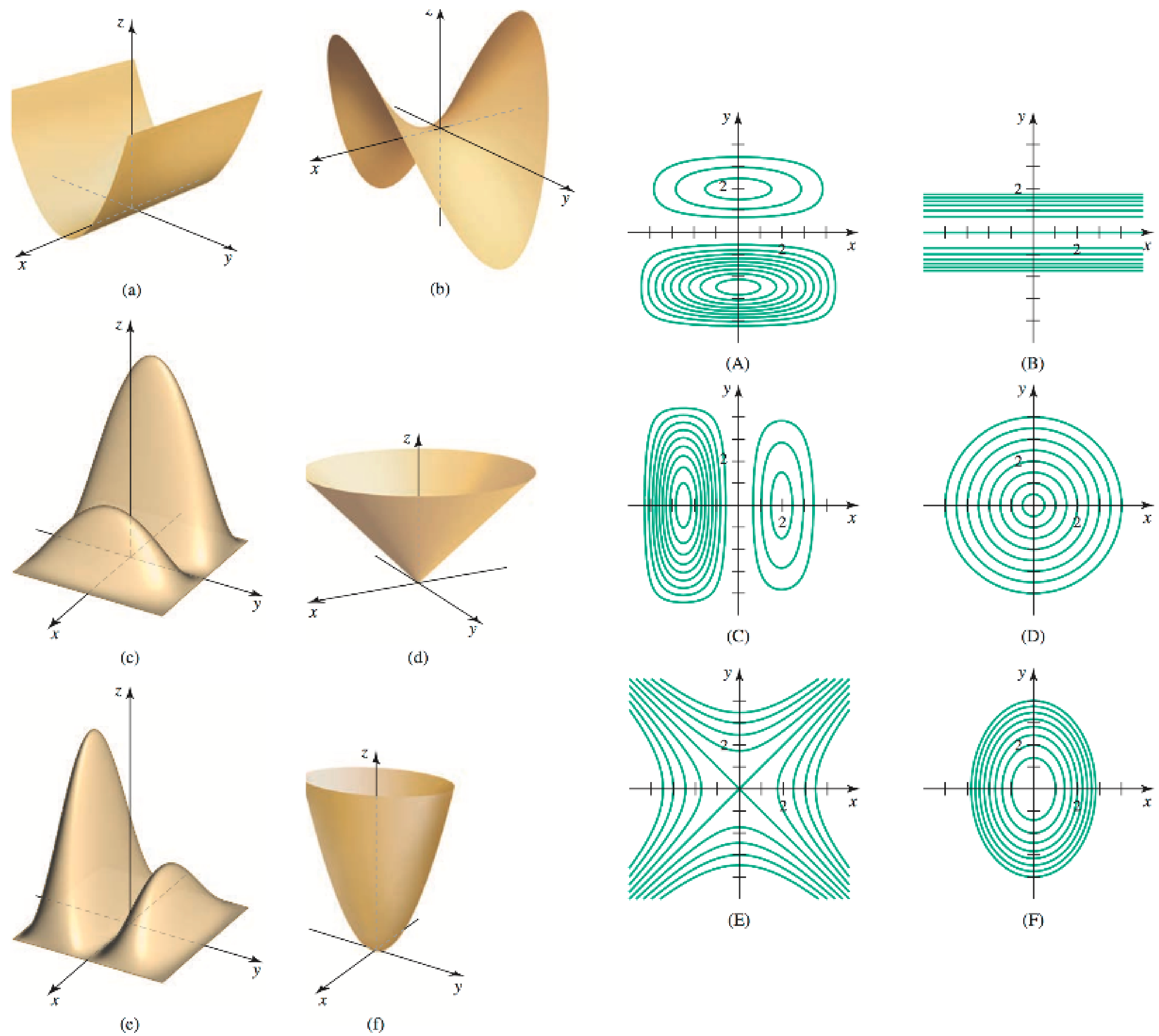
$$= \sqrt{(-t\sin(\frac{1}{2}t^2))^2 + (t\cos(\frac{1}{2}t^2)^2} = \sqrt{t^2(\cos^2(\frac{1}{2}t^2) + \sin^2(\frac{1}{2}t^2))} = \boxed{t}. \tag{20}$$

Since $\kappa(0) = 0$, we see that the clothoid and the linesegment have the same curvature at the point of transition, so the driver will not notice any jerking when switching from one part of the highway to the next.

Solution to (c): We were reminded in part (a) that a circle of radius r has a curvature of $\kappa = \frac{1}{r}$. It is clear from the preceding formula that $r = \frac{1}{\kappa}$, so the radius of a circle can also be determined using only its curvature. We see that $\kappa(1.2) = 1.2 = \frac{6}{5}$, so we want to use a circle with radius $r = \frac{1}{\frac{6}{5}} = \boxed{\frac{5}{6}}$ in order for the curvature of the circle to align with the curvature of the clothoid at $t = 1.2$ to ensure another smooth transition.

[1] We see that this is one of many instances in math in which it is useful to imagine that a line is just a circle of infinite radius.

Problem 4: Match surfaces a-f in the figure below with level curves A-F.



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a⇔(B): We see that the figure in a is similar to the cylinder generated by the curve $z = y^2$ in the yz-plane with the line $z = y = 0$, so the level sets will consist of lines that are parallel to the x-axis, which allows us to match a with (B).

b⇔(E): Since b is a hyperbolic paraboloid we know that its level sets are lines (for the level set of $z = 0$) and hyperbolas (for level sets of $z = z_0 \neq 0$), which allows us to match b with (E).

c⇔(C): The level sets of c are easily seen to be one or two (depending on the height of the level set) ellipse-like shapes whose major axes (by analogy, not literally) are parallel to the y-axis, which allows us to match c with (C).

d⇔(D): Since d is a cone we know that all of its level sets are circles, which allows us to match d with (D).

e⇔(A): The level sets of e are easily seen to be one or two (depending on the height of the level set) ellipse-like shapes whose major axes (by analogy, not literally) are parallel to the x-axis, which allows us to match e with (A).

f⇔(F): Since f is an elliptic paraboloid we know that all of its level sets are ellipses, which allws us to match f with (A).

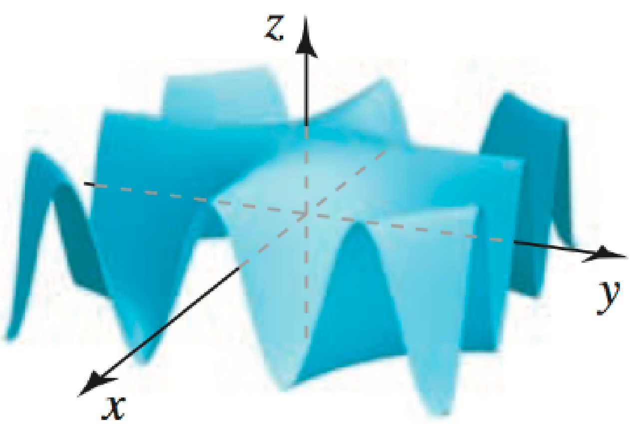
Problem 5: Match functions a-d with surfaces A-D in the figure below.

a. $f(x, y) = \cos(xy)$

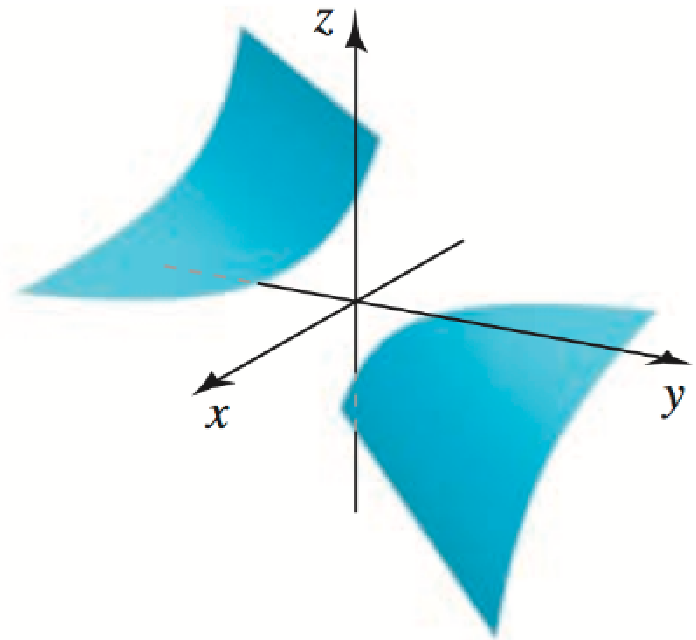
b. $g(x, y) = \ln(x^2 + y^2)$

c. $h(x, y) = \frac{1}{x-y}$

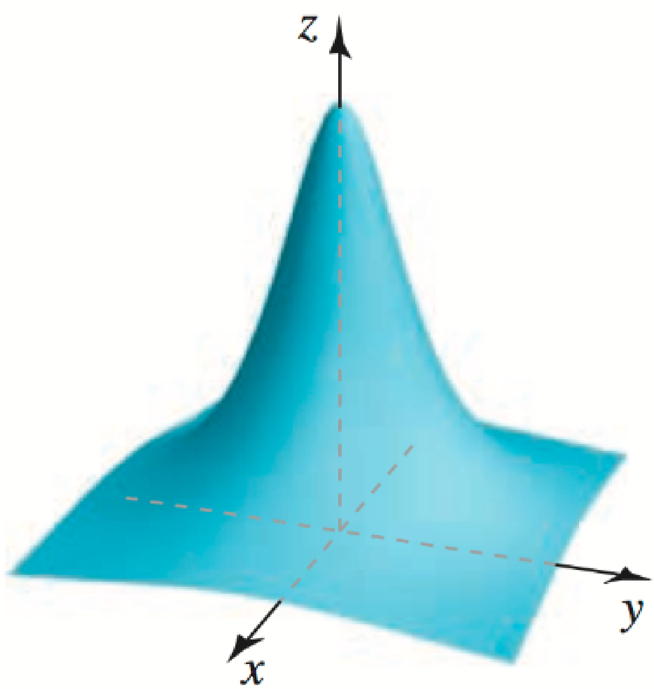
d. $p(x, y) = \frac{1}{1+x^2+y^2}$



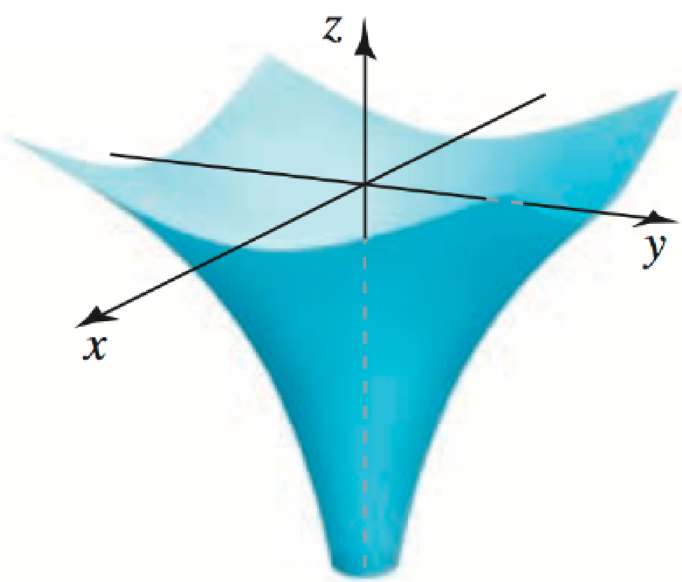
(A)



(B)



(C)



(D)

a⇔(A): We see that $1 = \cos(0\cdot y) = \cos(x\cdot 0)$ for all $x, y \in \mathbb{R}$, so f takes on the value 1 on the x-axis as well as the y-axis, which is enough to match a with (A). Alternatively, we may examine the behavior of f along some other line $y = mx$, i.e., examine $f(x, mx) = \cos(mx^2)$. One of the easiest such lines to examine is $y = x$, and we see that $f(x, x) = \cos(x^2)$ oscillates indefinitely, which is again enough information to yield the desired result.

b⇔(D): We begin by examining the level sets of $g(x, y)$. We see that $\ln(x^2 + y^2) = c \Leftrightarrow x^2 + y^2 = e^c$, so the level sets of $g(x, y)$ are circles centered at the origin. We also observe that $g(0, 0)$ is undefined and that $g(x, y)$ increases as $x^2 + y^2$ increases, so we deduce that b is matched with (D).

c⇔(B): We begin by examining the level sets of $h(x, y)$. We see that $\frac{1}{x-y} = c \Leftrightarrow x-y = \frac{1}{c}$, so the level sets of $h(x, y)$ are lines parallel to the line $x = y$, i.e., lines that make an angle of 45° with the positive x-axis. Furthermore, we see that if $x - y = c_1$ with $c_1 > 0$ then $h(x, y) > 0$, if $x - y = 0$ then $h(x, y)$ is undefined, and if $x - y = c_2$ with $c_2 < 0$ then $h(x, y) < 0$, which is enough information to match c with (B).

d⇔(C): We begin by examining the level sets of $p(x, y)$. We see that $\frac{1}{1+x^2+y^2} = c \Leftrightarrow x^2 + y^2 = \frac{1}{c} - 1$, so the level sets of $p(x, y)$ are circles centered at the origin. We observe that $p(x, y)$ is defined for all $x, y \in \mathbb{R}$ and that $p(x, y)$ decreases as $x^2 + y^2$ increases, so we deduce that d is matched with (C).