Problem 4: Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x,t) = 2\sin(\pi x)\sin(\frac{\pi}{2}t)$, for $0 \leq x \leq 1$ and $t \geq 0$ (see figure). At a fixed point in time, the string forms a wave on $[0,1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

(a) What is the period of the motion in time?

(b) Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).

(c) At a fixed time, what point on the string is moving fastest?

(d) At a fixed position on the string, when is the strong moving fastest?

(e) Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).

(f) At a fixed time, where is the slope of the string greatest?

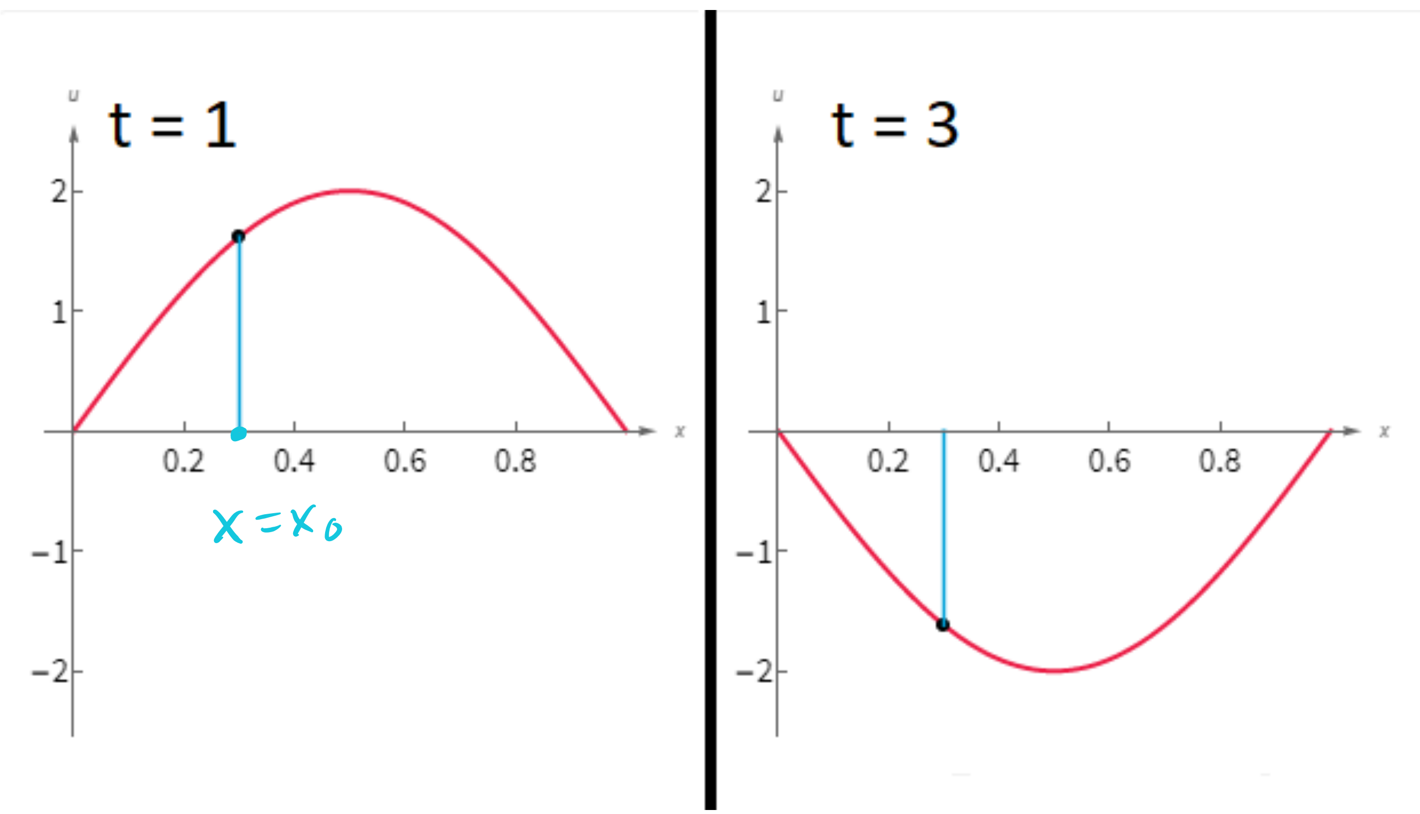

Figure 2: Snapshots of the wave at times $t = 1$ and $t = 3$.

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a) $\sin(y) = \sin(y + 2\pi)$ (sin is 2π-periodic)

What is the period p of $\sin\left(\frac{\pi}{2}t\right)$?

$$\sin\left(\frac{\pi}{2}(t+p)\right) = \sin\left(\frac{\pi}{2}t + 2\pi\right),$$

more specifically,

$$\frac{\pi}{2}t + \frac{\pi}{2}p = \frac{\pi}{2}t + 2\pi$$

$$\rightarrow \boxed{p = 4}$$

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b) $f(x_0, t)$ is the displacement function of the point $x = x_0$. So the velocity is

dell

$$v(x_0, t) = \frac{d}{dt} f(x_0, t) = \frac{\partial}{\partial t} f(x_0, t)$$

$$= f_t(x_0, t) := \lim_{h \to 0} \frac{f(x_0, t+h) - f(x_0, t)}{h}$$

$$f_t(x, t) = \lim_{h \to 0} \frac{f(x, t+h) - f(x, t)}{h}$$

$$= \frac{\partial}{\partial t} 2\sin(\pi x_0)\sin\left(\frac{\pi}{2}t\right)$$

recall that $\frac{d}{dx} 72e^{e^x+x^2} = 72\frac{d}{dx}e^{e^x+x^2}$

$$= 2\sin(\pi x_0)\frac{\partial}{\partial t}\sin\left(\frac{\pi}{2}t\right)$$

$$= 2\sin(\pi x_0)\left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)\right)$$

$$\boxed{v(\underset{1.5}{x_0},\underset{2}{t}) = \pi\sin(\pi x_0)\cos\left(\frac{\pi}{2}t\right)}$$

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c) $v(\underset{1.5}{x},\underset{2}{t}) = \pi\sin(\pi x)\cos\left(\frac{\pi}{2}t\right)$

If $t = t_0$, what $x \in [0,1]$ maximizes $v(x,t_0)$?

We optimize the 1-variable function

$$v(x,t_0) = \pi\sin(\pi x)\cos\left(\frac{\pi}{2}t_0\right).$$

$$\frac{d}{dx}v(x,t_0) = \underline{v_x(x,t_0)} = \frac{\partial}{\partial x}v(x,t_0)$$

(only for day 1)

$$= \frac{\partial}{\partial x} \pi \sin(\pi x) \cos\left(\frac{\pi}{2} t_0\right)$$

$$= \pi \cos\left(\frac{\pi}{2} t_0\right) \frac{\partial}{\partial x} \sin(\pi x)$$

$$= \pi \cos\left(\frac{\pi}{2} t_0\right) \left(\pi \cos(\pi x)\right)$$

$$= \pi^2 \cos\left(\frac{\pi}{2} t_0\right) \cos(\pi x) = \boxed{v_x(x, t_0)}$$

Now we find the critical points of $v(x, t_0)$ for $x \in [0,1]$.

$$0 = v_x(x, t_0) = \pi^2 \cos\left(\frac{\pi}{2} t_0\right) \cos(\pi x)$$

$\rightarrow \; 0 = \cos\left(\frac{\pi}{2} t_0\right) \cos(\pi x)$, if $\cos\left(\frac{\pi}{2} t_0\right) \neq 0$

then $0 = \cos(\pi x) \rightarrow \pi x = \frac{\pi}{2} + \pi y, \; y \in \mathbb{Z}$

$x = \ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \rightarrow x = \frac{1}{2}$ b/c $x \in [0,1]$.

So $x=\frac{1}{2}$ is the only C.P., so

we compare $v(\frac{1}{2},t_0)$, $v(0,t_0)$, and $v(1,t_0)$.

$v(x,t_0)=\pi\sin(\pi x)\cos(\frac{\pi}{2}t_0)$.

$\rightarrow v(0,t_0)=v(1,t_0)=0$,

$$v(\tfrac{1}{2},t_0)=\pi\sin(\tfrac{\pi}{2})\cos(\tfrac{\pi}{2}t_0)$$
$$=\pi\cos(\tfrac{\pi}{2}t_0)$$

$\rightarrow$ Speed at $x=\frac{1}{2}$ when $t=t_0$

is $\left|\pi\underbrace{\cos(\tfrac{\pi}{2}t_0)}_{\neq 0}\right|>0$,

So $\boxed{x=\frac{1}{2}}$ has the largest speed.

if $\cos(\frac{\pi}{2}t_0)=0$, then

$v(x,t_0)=\pi\sin(\pi x)\cos(\frac{\pi}{2}t_0)=0$,

so $\boxed{\text{all } x\in[0,1]}$ have a speed of 0.

$\cos\left(\frac{\pi}{2}t_0\right) = 0$ iff $\frac{\pi}{2}t_0 = \frac{\pi}{2} + y\pi, y \in \mathbb{Z}$

$\rightarrow$ $t = \underbrace{2n+1}_{}, n \in \mathbb{N}$ $(t > 0)$

positive odd integer times.

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d) We again fix $x = x_0$. We want to optimize

Speed $= |v(x_0, t)|$, so we start by optimizing velocity $v(x_0, t)$ since $\text{Speed}_{max} = \max(v_{max}, |v_{min}|)$.

We start by finding the C.P.s, so when

$$0 = \frac{\partial}{\partial t} v(x_0, t) = \frac{\partial}{\partial t} \pi \sin(\pi x_0) \cos\left(\frac{\pi}{2}t\right)$$

using Calc 1 reasoning

$$= \pi \sin(\pi x_0) \frac{\partial}{\partial t} \cos\left(\frac{\pi}{2}t\right)$$

$$= \pi \sin(\pi x_0) \left(-\frac{\pi}{2} \sin\left(\frac{\pi}{2}t\right)\right)$$

$= -\frac{\pi^2}{2} \sin(\pi x_0) \sin(\frac{\pi}{2} t)$

$\rightarrow 0 = \sin(\pi x_0) \sin(\frac{\pi}{2} t)$

We assume that $x_0 \neq 0, 1$ b/c

$v(0,t) = v(1,t) = 0$ for all $t \geq 0$.

Now $\sin(\pi x_0) \neq 0$ for $x_0 \in (0,1)$,

So we divide by $\sin(\pi x_0)$.

$\rightarrow 0 = \sin(\frac{\pi}{2} t) \rightarrow \frac{\pi}{2} t = \pi y, y \in \mathbb{Z}$

$\rightarrow t = \underbrace{2z}_{\text{even integers}}, z \in \mathbb{Z} \rightarrow t = \underbrace{2n}_{\text{positive even integers and } 0}, n \in \mathbb{N} \cup \{0\}$ b/c $t \geq 0$,

So we compare the values

of $v(x_0, 2n)$, $n \in \mathbb{N} \cup \{0\}$

C.P. and endpoint by coincidence

$$v(x_0, 2n) = \pi \sin(\pi x_0) \cos\left(\frac{\pi}{2} \cdot 2n\right)$$

$$= \pi \sin(\pi x_0) \underline{\cos(n\pi)}$$

$$= \underline{(-1)}^n \pi \sin(\pi x_0)$$

$\rightarrow$ Speed $(x_0, 2n)$

$$= |v(x_0, 2n)| = |\pi \sin(\pi x_0)|$$

$$= \boxed{\pi |\sin(\pi x_0)|} \quad \forall n \in \mathbb{N} \cup \{0\}$$

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e) $f(x, t_0)$ represents the displacement at position x during time $t = t_0$

The rate of change of displacement w.r.t. position is

$$\frac{d}{dx} f(x, t_0) = \frac{\partial}{\partial x} f(x, t_0) = f_x(x, t_0)$$

$\uparrow$

(only for day 2) $= \lim_{h \to 0} \frac{f(x+h, t_0) - f(x, t_0)}{h}$

$$\frac{\partial}{\partial x} f(x,t) = f_x(x,t) = \lim_{h \to 0} \frac{f(x+h,t) - f(x,t)}{h}$$

$$= \frac{\partial}{\partial x} 2\sin(\pi x)\sin(\tfrac{\pi}{2} t_0)$$

$$\frac{d}{dx} 17 e^{e^x + x^e} = 17 \frac{d}{dx} e^{e^x + x^e}$$

$$= 2\sin(\tfrac{\pi}{2} t_0) \frac{\partial}{\partial x} \sin(\pi x)$$

$$= 2\sin(\tfrac{\pi}{2} t_0)\left(\pi \cos(\pi x)\right)$$

$$= \boxed{2\pi \sin(\tfrac{\pi}{2} t_0)\cos(\pi x)}$$

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f) $f_x(x,t) = 2\pi \underline{\sin(\tfrac{\pi}{2} t)}\cos(\pi x)$ $\rightarrow$ $f_x(x,t_0) = 0$ $\forall x$

is the slope of the string at time t and position x.

We want to optimize $f_x(x,t_0)$ with respect to x.

→ We begin by finding the critical points of $f_x(x,t)$.

$$0 = \frac{\partial}{\partial x} f_x(x,t)$$

$$= \frac{\partial}{\partial x} 2\pi \cos(\pi x) \sin(\tfrac{\pi}{2} t_0)$$

as before

$$= 2\pi \sin(\tfrac{\pi}{2} t_0) \frac{\partial}{\partial x} \cos(\pi x)$$

$$= 2\pi \sin(\tfrac{\pi}{2} t_0)(-\pi \sin(\pi x))$$

$$= -2\pi^2 \sin(\tfrac{\pi}{2} t_0) \sin(\pi x)$$

$$\rightarrow 0 = \sin(\tfrac{\pi}{2} t_0) \sin(\pi x)$$

if $\sin(\frac{\pi}{2} t_0) \neq 0$, then divide by it

$\rightarrow 0 = \sin(\pi x) \Rightarrow \pi x = \pi y,\ y \in \mathbb{Z}$

$\rightarrow x = 0, 1$ b/c $x \in [0,1]$.

Coincidentally, C.P.s and endpoints are the same.

$$f_x(0, t_0) = 2\pi \sin(\tfrac{\pi}{2} t_0)\cos(0)$$
$$= 2\pi \sin(\tfrac{\pi}{2} t_0)$$

$$f_x(1, t_0) = 2\pi \sin(\tfrac{\pi}{2} t_0)\cos(\pi)$$
$$= -2\pi \sin(\tfrac{\pi}{2} t_0)$$

else

if $\sin(\tfrac{\pi}{2} t_0) \neq 0$

$$\rightarrow \left|2\pi \sin(\tfrac{\pi}{2} t_0)\right| = 2\pi \left|\sin(\tfrac{\pi}{2} t_0)\right|$$