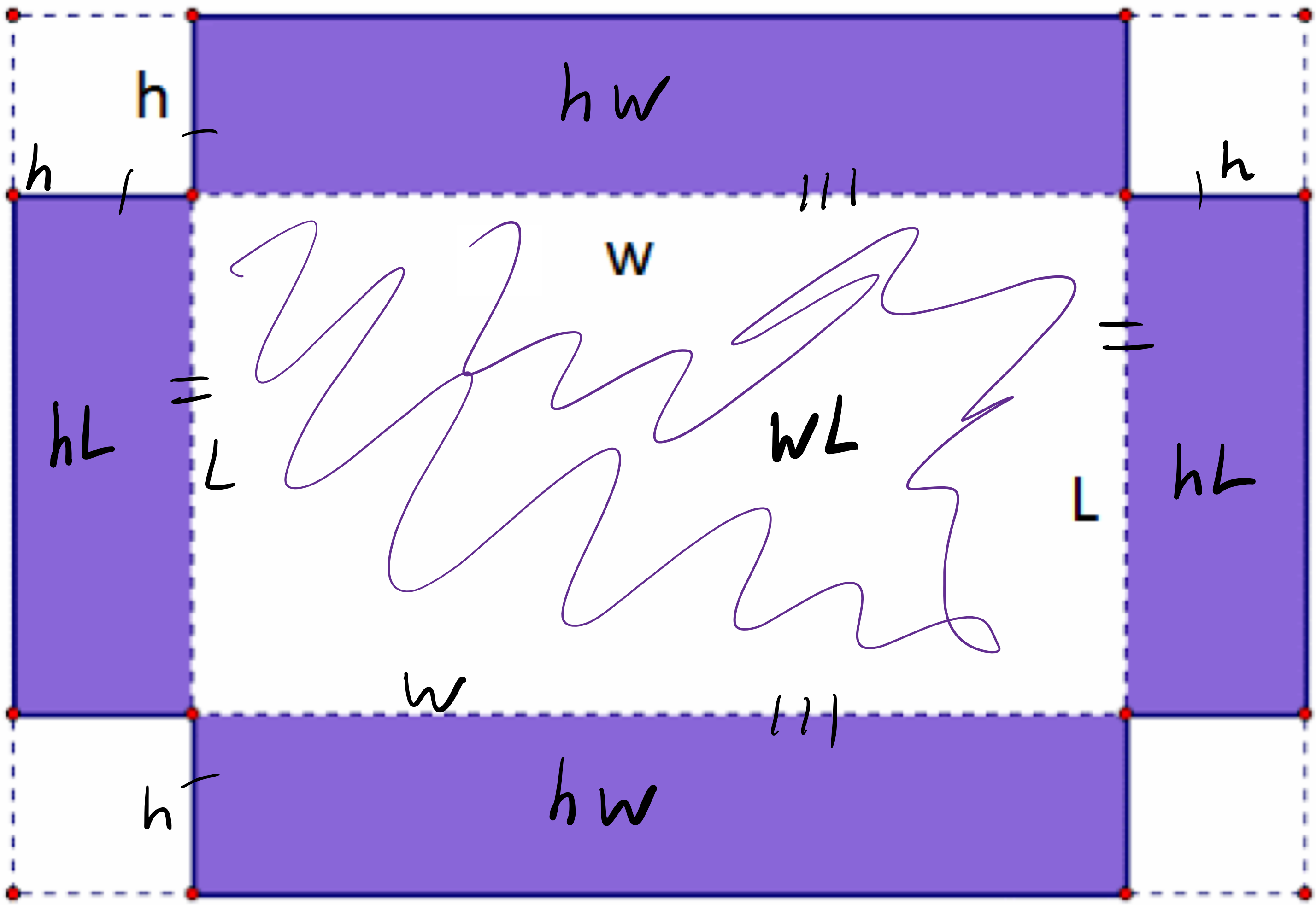
Problem 2: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.



$S = S(h,w,L) = \text{Surface area} = 2hw + 2hL + wL$

$4 = \text{Volume} = hwL \rightarrow h = \frac{4}{wL}$

$S = S(w,L) = S(h(w,L), w, L)$

$= 2\left(\frac{4}{wL}\right)w + 2\left(\frac{4}{wL}\right)L + wL$

$= \frac{8}{L} + \frac{8}{w} + wL, \quad w, L > 0$

We start by finding a <u>local</u> min.

$$0 = S_w \qquad 0 = -\frac{8}{w^2} + L$$

$$\longleftrightarrow$$

$$0 = S_L \qquad 0 = -\frac{8}{L^2} + W$$

$$\longleftrightarrow L = \frac{8}{w^2} \to W = \frac{8}{\left(\frac{8}{w^2}\right)^2} = \frac{8}{\frac{64}{w^4}}$$

$$W = \frac{8}{L^2}$$

$$= \frac{w^4}{8} \quad (w \neq 0)$$

$$L = \frac{8}{2^2} = 2$$

$$\to 1 = \frac{w^3}{8} \to w^3 = 8$$

$$\to \boxed{w = 2}$$

$\to (w, L) = (2, 2)$

is the only C.P.

$$S_{LL} = \frac{\partial}{\partial L} S_L = \frac{\partial}{L}\left(-\frac{8}{L^2} + w\right) = \frac{16}{L^3} + 0$$

$$S_{ww} = \frac{\partial}{\partial w} S_w = \frac{\partial}{\partial w}\left(-\frac{8}{w^2} + L\right) = \frac{16}{w^3}$$

$$S_{LW} = S_{WL} = \frac{\partial}{\partial w} S_L = \frac{\partial}{\partial w}\left(-\frac{8}{L^2} + w\right) = 1$$

(Clairut's Thm)

$$D(W,L) = S_{LL} S_{WW} - (S_{WL})^2$$

$$= \frac{16}{L^3} \cdot \frac{16}{W^3} - 1^2 = \frac{256}{L^3 W^3} - 1$$

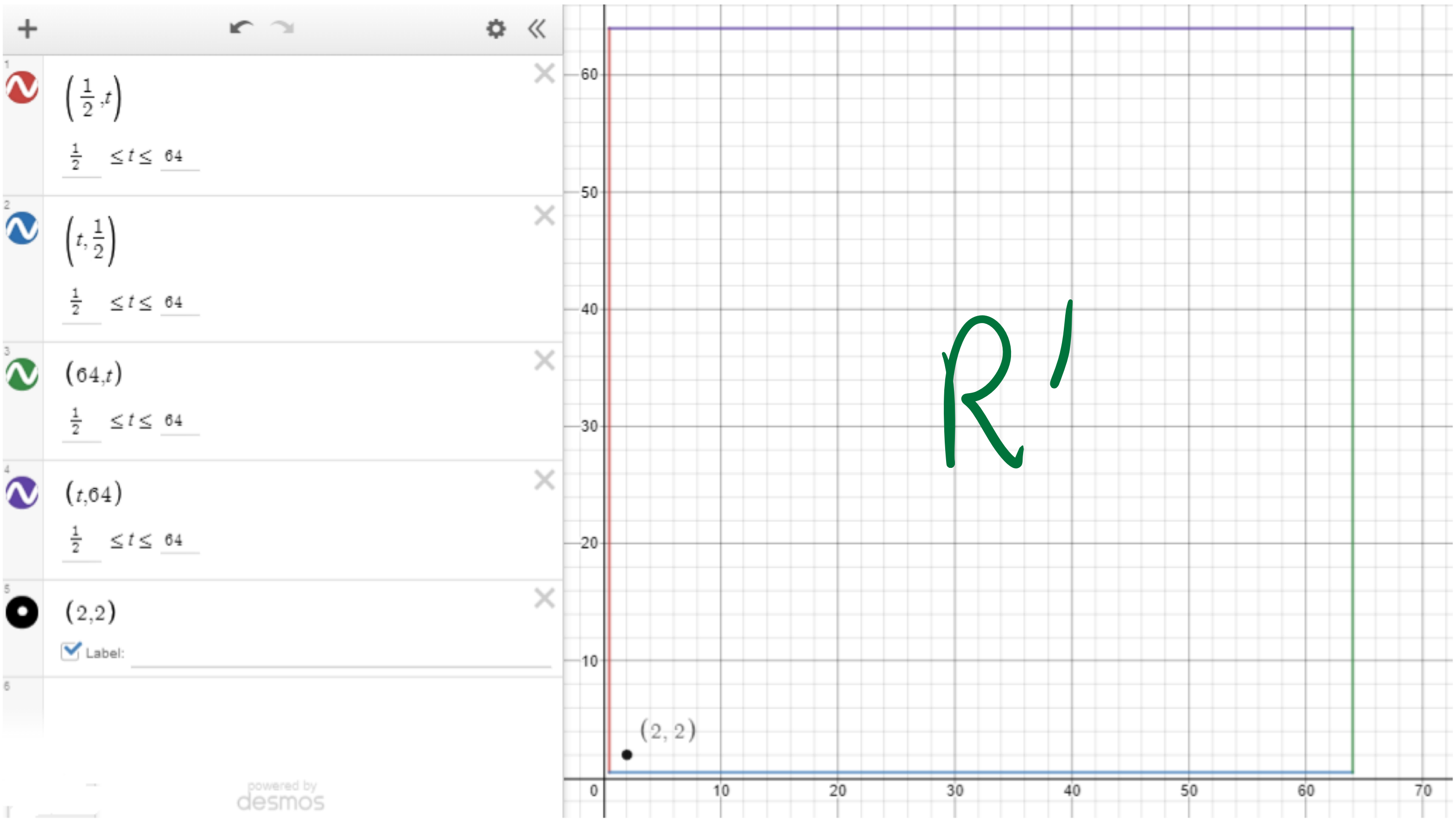
$$D(2,2) = \frac{256}{2^3 \cdot 2^3} - 1 = \frac{256}{64} - 1 = 4 - 1$$

$$= 3 > 0$$

$$f_{WW}(2,2) = \frac{-8}{2^3} + 2 = 1 > 0$$

(or check $f_{LL}(2,2)$) $\rightarrow$ (2,2) is a local min.

Now let's show the global min is at (2,2) by using EVT.



$S(w,L) = \frac{8}{L} + \frac{8}{w} + wL$, $S(2,2)=12$

If $L < \frac{1}{2}$ or $w < \frac{1}{2}$, then $S(w,L) > 16 > 12$.

If $L \geq \frac{1}{2}$ and $w \geq 64$, then $S(w,L) \geq 32 > 12$

If $w \geq \frac{1}{2}$ and $L \geq 64$, then $S(w,L) \geq 32$.

We have now shown that $S(w,L) > 16$ for $(w,L) \notin R'$, and $S(w,L) \geq 16$ for $(w,L) \in \partial R'$.

Since $(2,2)$ is the only C.P. of S in R', which is a closed and bounded region on which S is continuous, the EVT shows us that the global min occurs at $(2,2)$.

Problem 3: Consider the function $f(x, y) = 3 + x^4 + 3y^4$. Show that $(0, 0)$ is a critical point for $f(x, y)$ and show that the second derivative test is inconclusive at $(0, 0)$. Then describe the behavior of $f(x, y)$ at $(0, 0)$.
*H*int: The product of 2 negative numbers is positive.

$f_x(x,y) = 4x^3$ $\quad 0 = f_x \quad 0 = 4x^3$

$f_y(x,y) = 12y^3$, $\quad 0 = f_y \leftrightarrow 0 = 12y^3$

$\rightarrow (x,y) = (0,0)$ is the only C.P.

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(4x^3) = 12x^2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(12y^3) = 36y^2$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(4x^3) = 0$$

(Clairut's Thm will hold for all f in this course)

$$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$$
$$= 12x^2 \cdot 36y^2 - 0^2$$
$$= 432 x^2 y^2$$

$\rightarrow D(0,0) = 0 \rightarrow$ the Second Derivative Test is inconclusive.

$x^4 = (x^2)^2 \geq 0$ and $x^4 = 0$ iff $x = 0$

$y^4 = (y^2)^2 \geq 0$ and $y^4 = 0$ iff $y = 0$,

so $3 + x^4 + y^4 \geq 3$ and

$3 + x^4 + y^4 = 3$ iff $x = y = 0$, so

$(x, y) = (0, 0)$ is where f attains

its absolute minimum value.

Problem 6: Find the absolute minimum and absolute maximum values of the function

$$f(x, y) = \underbrace{x^2 + 4y^2}_{0 \le \qquad \le} + 1 \tag{1}$$

over the region

$$R = \{(x, y) : x^2 + 4y^2 \le 1\}. \tag{2}$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

R is a closed and bounded region on which f is continuous, so by EVT f attains its absolute minimum and absolute maximum values on R.

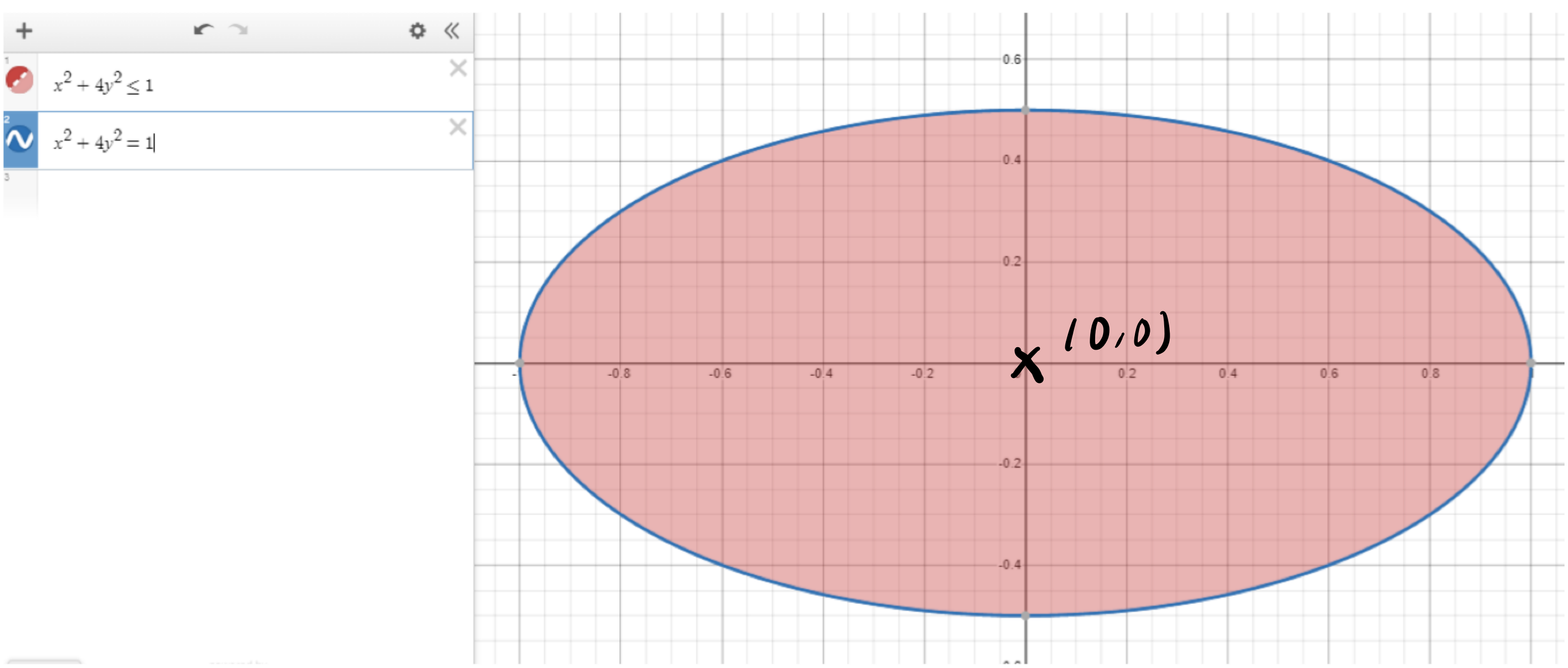


Figure 2: The interior of the R is shaded in red and the boundary of R is blue.

$f(x,y) = x^2 + 4y^2 + 1$, $f_x(x,y) = 2x$, $f_y(x,y) = 8y$

$$\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix} \iff \begin{matrix} 2x = 0 \\ 8y = 0 \end{matrix} \iff (x,y) = (0,0),$$

So $(0,0)$ is the only C.P. of f, and it is inside R.

$$\boxed{f(0,0) = 0^2 + 4 \cdot 0^2 + 1 = 1}$$

To find the extreme values of f on $\underbrace{\partial R}_{\text{Boundary of } R}$ we begin by parametrizing ∂R.

$\vec{r}(t) = \langle \cos(t), \frac{1}{2}\sin(t) \rangle$, $0 \le t \le 2\pi$

is a parametrization of ∂R.

$$h(t) = f(\vec{r}(t)) = f(\cos(t), \tfrac{1}{2}\sin(t))$$

$$= \cos^2(t) + 4\left(\tfrac{1}{2}\sin(t)\right)^2 + 1$$

$$= \cos^2(t) + \sin^2(t) + 1 = 2 \Rightarrow$$

$f(x,y) = 2$ for <u>all</u> $(x,y) \in \partial R$.

$\hookrightarrow f(0,0)=1$

The absolute min. value of f is 1, and it is attained at $(0,0)$.

The absolute max. value of f is 2, and it is ~~at~~ attained at <u>every</u> $(x,y)\in\partial R$.

Problem 8: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

$$f(x, y, z) = xyz \tag{5}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{6}$$

$g(x,y,z) = x^2+2y^2+4z^2-9$ is the constraint function.

$$\vec{\nabla} f(x,y,z) = \langle f_x, f_y, f_z \rangle$$
$$= \langle yz, xz, xy \rangle$$

$$\vec{\nabla} g(x,y,z) = \langle g_x, g_y, g_z \rangle$$
$$= \langle 2x, 4y, 8z \rangle$$

The method of Lagrange multipliers gives us the system

$$g(x,y,z) = 0$$
$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

$$\Leftrightarrow \quad x^2+2y^2+4z^2-9=0$$

$$yz = 2\lambda x$$

$$xz = 4\lambda y$$

$$xy = 8\lambda z$$

By cross-multiplication

$$\underline{8\lambda xz^2} = \underline{4\lambda xy^2}$$

$$8\lambda xz^2 - 4\lambda xy^2 = 0$$

$$2\lambda xz^2 - \lambda xy^2 = 0$$

$$\underline{\lambda}\,\underline{x}\,\underline{(2z^2-y^2)} = 0$$

By the zero-product property

If $ab=0 \rightarrow a=0$ or $b=0$

(Similarly, if $abc=0 \rightarrow a=0$, or $b=0$, or $c=0$)

We have 3 cases.

Case 1, $\lambda = 0$:

$$x^2 + 2y^2 + 4z^2 - 9 = 0$$

$$\left.\begin{aligned} yz &= 0 \\ xy &= 0 \\ xz &= 0 \end{aligned}\right\} \rightarrow (x, y, z) \in \{(x,0,0), (0,y,0), (0,0,z)\}$$

$\rightarrow$ either $x^2 + 0 + 0 - 9 = 0$ or

$0 + 2y^2 + 0 - 9 = 0$ or

$0 + 0 + 4z^2 - 9 = 0$

$$\rightarrow \left\{(3,0,0), (-3,0,0), \left(0, \tfrac{3}{\sqrt{2}}, 0\right), \left(0, -\tfrac{3}{\sqrt{2}}, 0\right), \left(0,0,\tfrac{3}{2}\right), \left(0,0,-\tfrac{3}{2}\right)\right\}$$

Case 2, $x = 0$ (and $\lambda \neq 0$):

$$2y^2 + 4z^2 - 9 = 0$$

$$yz = 0$$

$$y = 0 \Leftarrow 0 = 4\lambda y$$

$$z = 0 \Leftarrow 0 = 8\lambda z$$

$\Rightarrow (0,0,0)$ is Not a C.P. because $g(0,0,0) = 0 \Rightarrow$

No C.P.s from this case.

Case 3 ......