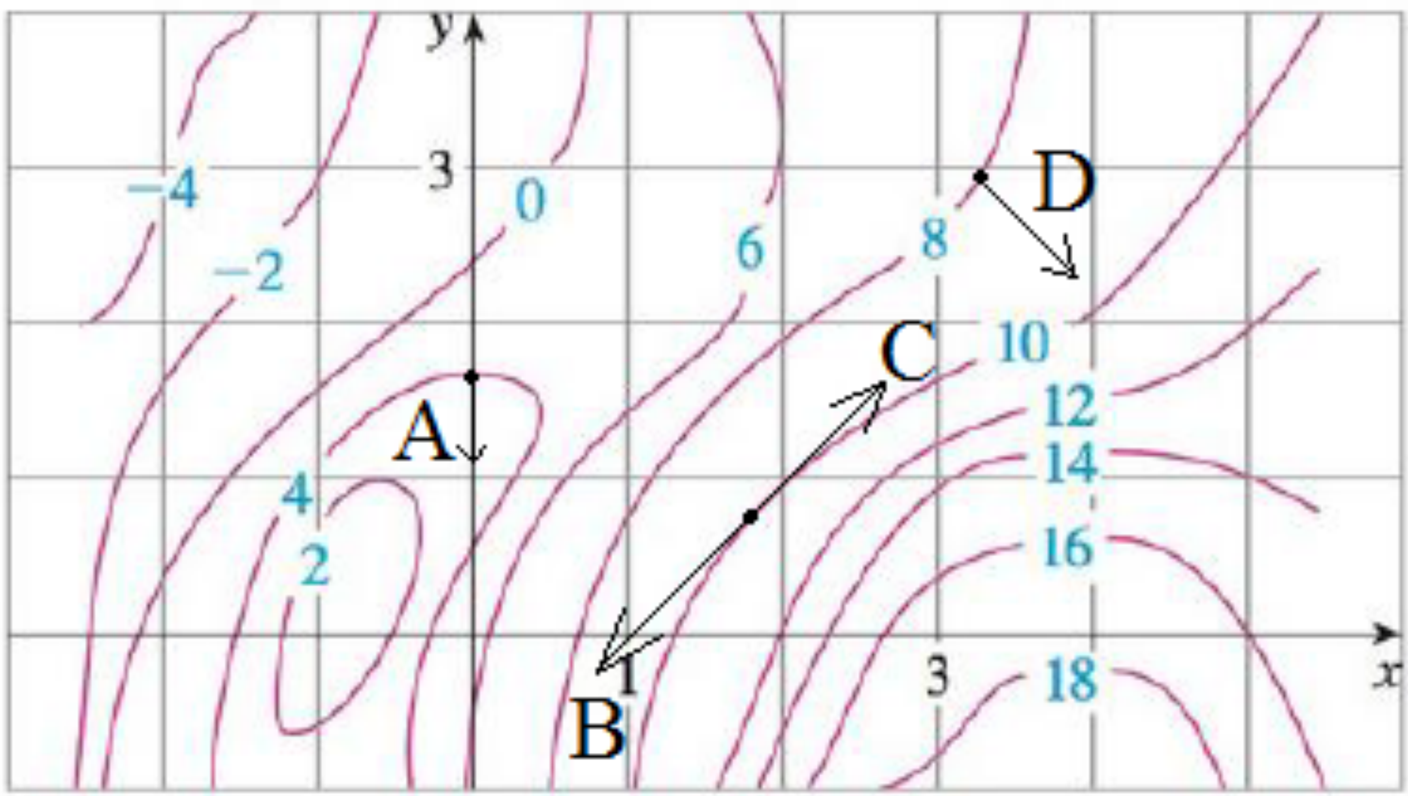
Problem 1: Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.



B, C point in the direction of no change

Figure 1: Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

~~(A)~~ ~~(B)~~ ~~(C)~~ (D) (E) None of the given vectors

A could be $-\vec{\nabla} f$

1) Gradient of f is $\perp$ to the level curves of f.

$\boxed{2)}$ Gradient of f points in the direction of maximum increase for f.

$-\vec{\nabla} f$ points in the direction of maximum decrease for f

Problem 2: Consider the function $f(x, y) = x^2 + y^2$ and and the point $P = (2, 3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.

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a) We $\dfrac{-\vec{\nabla} f(P)}{\|-\vec{\nabla} f(P)\|}$

$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle$

$-\vec{\nabla} f(P) = -\vec{\nabla} f(2,3) = \langle -4, -6 \rangle$

$\|-\vec{\nabla} f(P)\| = \sqrt{(-4)^2 + (-6)^2} = \sqrt{52}$

$= 2\sqrt{13}$

$\dfrac{-\vec{\nabla} f(P)}{\|-\vec{\nabla} f(P)\|} = \dfrac{\langle -4, -6 \rangle}{2\sqrt{13}} = \langle -\dfrac{2}{\sqrt{13}}, -\dfrac{3}{\sqrt{13}} \rangle$

b) $D_{\hat{u}} f(P) = \vec{\nabla} f(P) \cdot \hat{u}$

($\hat{u}$ has to be a unit vector).

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle 3,2 \rangle}{\sqrt{3^2+2^2}} = \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$\vec{\nabla} f(P) = \langle 4, 6 \rangle$$

$$D_{\hat{u}} f(P) = \langle 4, 6 \rangle \cdot \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$= \frac{12}{\sqrt{13}} + \frac{12}{\sqrt{13}} = \frac{24}{\sqrt{13}}$$

Problem 4: Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function

$$\varphi(x,y) = \sqrt{x^2+y^2}, \quad \text{for } x^2+y^2 \leq 9, (x,y) \neq (0,0). \tag{1}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

$$\frac{\partial}{\partial x}\varphi(x,y) = \frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{x^2+y^2}}$$, and similarly

$$\frac{\partial}{\partial y}\varphi(x,y) = \frac{y}{\sqrt{x^2+y^2}} \rightarrow$$

$$\vec{F}(x,y) = \vec{\nabla}\varphi(x,y) = \langle \varphi_x, \varphi_y \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\|\vec{\nabla}\varphi(x,y)\| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}$$

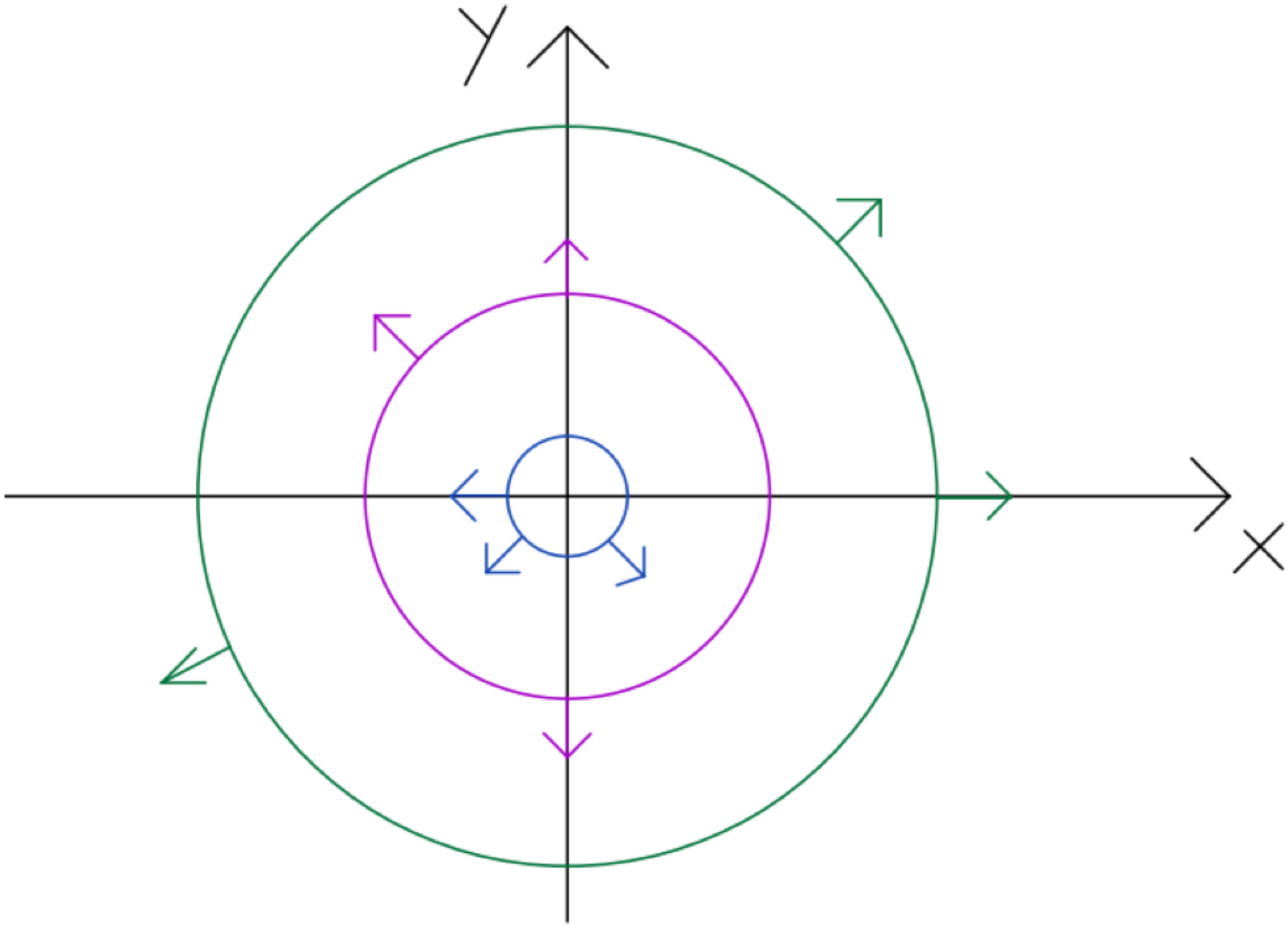
$$= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}$$

$$= \sqrt{1} = 1$$

$$\varphi(x,y) = C \rightarrow$$

$$\sqrt{x^2+y^2} = C$$

$$x^2+y^2 = c^2$$


Y
X

Problem 6: Let $w = f(x, y, z) = 2x + 3y + 4z$, which is defined for all $(x, y, z) \in \mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

(a) We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = f(x, y, z(x, y))$. Differentiate with respect to x holding y fixed to show that $\left(\frac{\partial w}{\partial x}\right)_y = 18$, where the subscript y indicates that y is held fixed.

(b) Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = f(x, y(x, z), z)$ and differentiate with respect to x holding z fixed. Show that $\left(\frac{\partial w}{\partial x}\right)_z = 8$, where the subscript z indicates that z is held fixed.

(c) Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

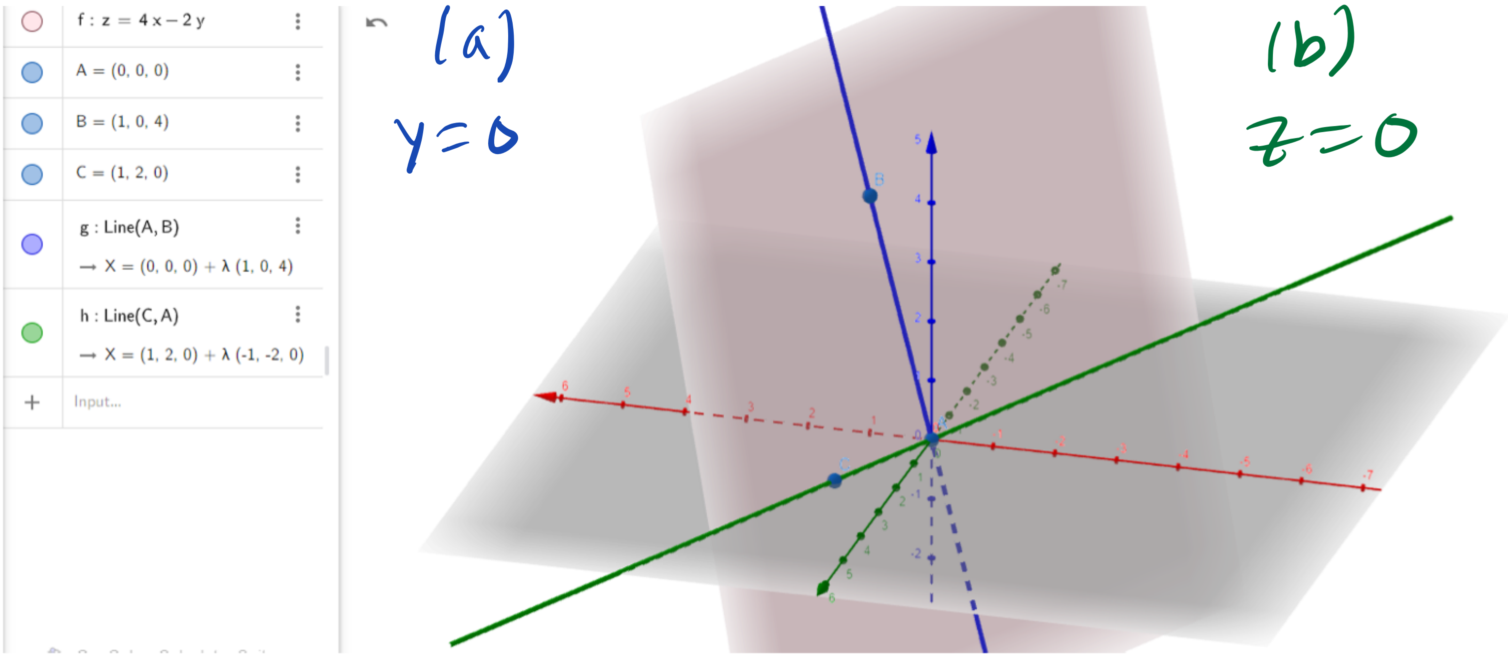
$$z(x,y) = 4x - 2y, \quad f(x,y,z(x,y)) = 2x + 3y + 4(4x - 2y)$$
$$= 18x - 5y$$

$$\rightarrow \frac{\partial}{\partial x} w = 18$$

$$z = 4x - 2y \rightarrow 2y = 4x - z \rightarrow y = y(x,z) = 2x - \tfrac{1}{2} z$$

$$f(x, y(x,z), z) = 2x + 3\left(2x - \tfrac{1}{2} z\right) + 4z$$
$$= 8x + \tfrac{5}{2} z$$

$$\rightarrow \frac{\partial}{\partial x} w = 8$$



$$\frac{\partial}{\partial x} w(0,0,0)$$

$$\frac{\partial}{\partial x} w(0,0,0) = D_{\langle \frac{1}{\sqrt{17}}, 0, \frac{4}{\sqrt{17}} \rangle} w(0,0,0)$$

$$\frac{\partial}{\partial x} w(0,0,0) = D_{\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0 \rangle} w(0,0,0)$$