Problem 1: Match function a-f with the appropriate graph A-F.

a. $\vec{r}(t) = \langle t, -t, t \rangle$.

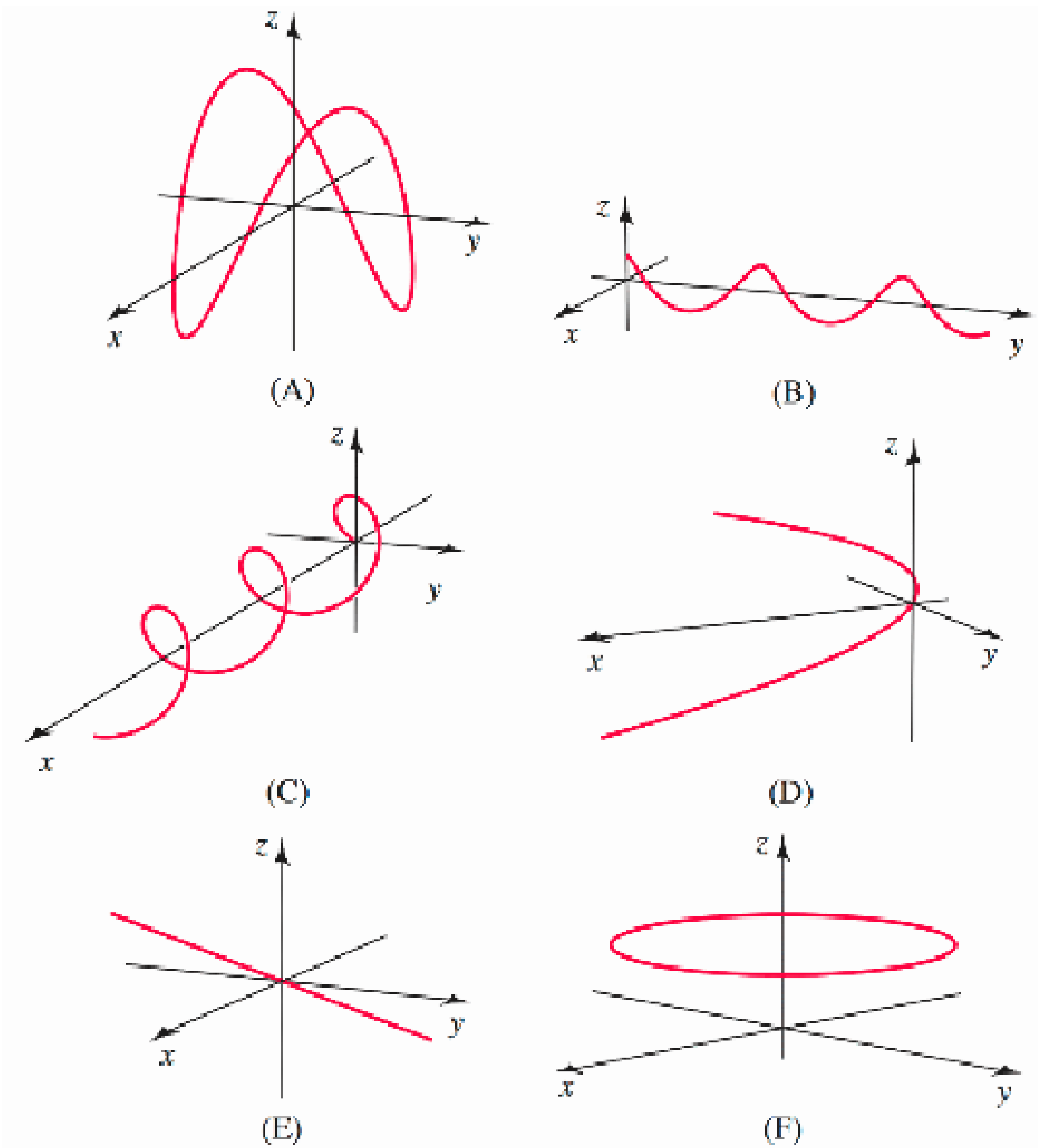
b. $\vec{r}(t) = \langle t^2, t, t \rangle$.

c. $\vec{r}(t) = \langle 4\cos(t), 4\sin(t), 2 \rangle$.

d. $\vec{r}(t) = \langle 2t, \sin(t), \cos(t) \rangle$.

e. $\vec{r}(t) = \langle \sin(t), \cos(t), \sin(2t) \rangle$.

f. $\vec{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle$.



Solution:

a $\leftrightarrow$ E It is clear that $\vec{r}(t)$ is the paramterization of a straight line and (E) is the only graph of a straight line in the available options.

b $\leftrightarrow$ D We see that $\vec{r}(t)$ is a parabola if we make a plane with the line $y = z$ taking the place of (what is normally) the x-axis and the line $y = z = 0$ taking the place of (what is normally) the y-axis and (D) is the only graph of a parabola.

c $\leftrightarrow$ F We see that the z-coordinate of $\vec{r}(t)$ is constant so the graph of $\vec{r}(t)$ lies in a horizontal plane and (F) is the only such graph.

d $\leftrightarrow$ C We see that when the x-coordinate of $\vec{r}(t)$ is ignored the result is a paramterization of the unit circle in the yz-plane, so if the graph of $\vec{r}(t)$ is "smushed down to the yz-plane" then the result will be the unit circle. Furthermore, it is clear that the x-coordinate of $\vec{r}(t)$ is unbounded and (C) is the only graph that satisfies the previous two properties.

e $\leftrightarrow$ A We see that all three components of $\vec{r}(t)$ are bounded and that none of the components are constant and (A) is the only graph that satisfies these properties.

f $\leftrightarrow$ B We see that when the y-coordinate of $\vec{r}(t)$ is ignored the result is a parameterization of the unit circle in the xz-plane, so if the graph of $\vec{r}(t)$ is "smushed down to the xz-plane" then the result will be the unit circle. Furthermore, it is clear that the y-coordinate of $\vec{r}(t)$ is unbounded and (B) is the only graph that satisfies the previous two properties.

Problem 2: Find an equation of the plane P through the points R$(5,3,7)$, S$(0,1,0)$, and T$(1,2,1)$.

Solution: It will be relatively easy to find the equation of the plane P if we first find a vector $\vec{n}$ that is normal to P. To find such a $\vec{n}$ it suffices to take the cross product of any two nonparallel vectors lying in P. To this end, we see that

$$\overrightarrow{\mathrm{SR}} = \langle 5,3,7\rangle - \langle 0,1,0\rangle = \langle 5,2,7\rangle, \text{ and} \tag{1}$$

$$\overrightarrow{\mathrm{ST}} = \langle 1,2,1\rangle - \langle 0,1,0\rangle = \langle 1,1,1\rangle \tag{2}$$

are two nonparallel vectors lying in P. We now take

$$\vec{n} = \overrightarrow{\mathrm{SR}} \times \overrightarrow{\mathrm{ST}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & 7 \\ 1 & 1 & 1 \end{vmatrix} \tag{3}$$

$$\hat{i}(2\cdot 1 - 1\cdot 7) - \hat{j}(5\cdot 1 - 1\cdot 7) + \hat{k}(5\cdot 1 - 1\cdot 2) \tag{4}$$

$$= -5\hat{i} + 2\hat{j} + 3\hat{k} = \langle -5,2,3\rangle. \tag{5}$$

To derive the equation of the plane P we recall that $\vec{n}$ is perpendicular to any vector that lies in P. It follows that if (x,y,z) is an arbitrary point in P, then since $(0,1,0)$ is also a point in P the vector

$$\vec{v} := \langle x,y,z\rangle - \langle 0,1,0\rangle = \langle x, y-1, z\rangle \tag{6}$$

is a vector contained in P, so we have

$$0 = \vec{n}\cdot\vec{v} = \langle -5,2,3\rangle \cdot \langle x,y-1,z\rangle = -5x + 2y - 2 + 3z \tag{7}$$

$$\to \boxed{2 = -5x + 2y + 3z}. \tag{8}$$

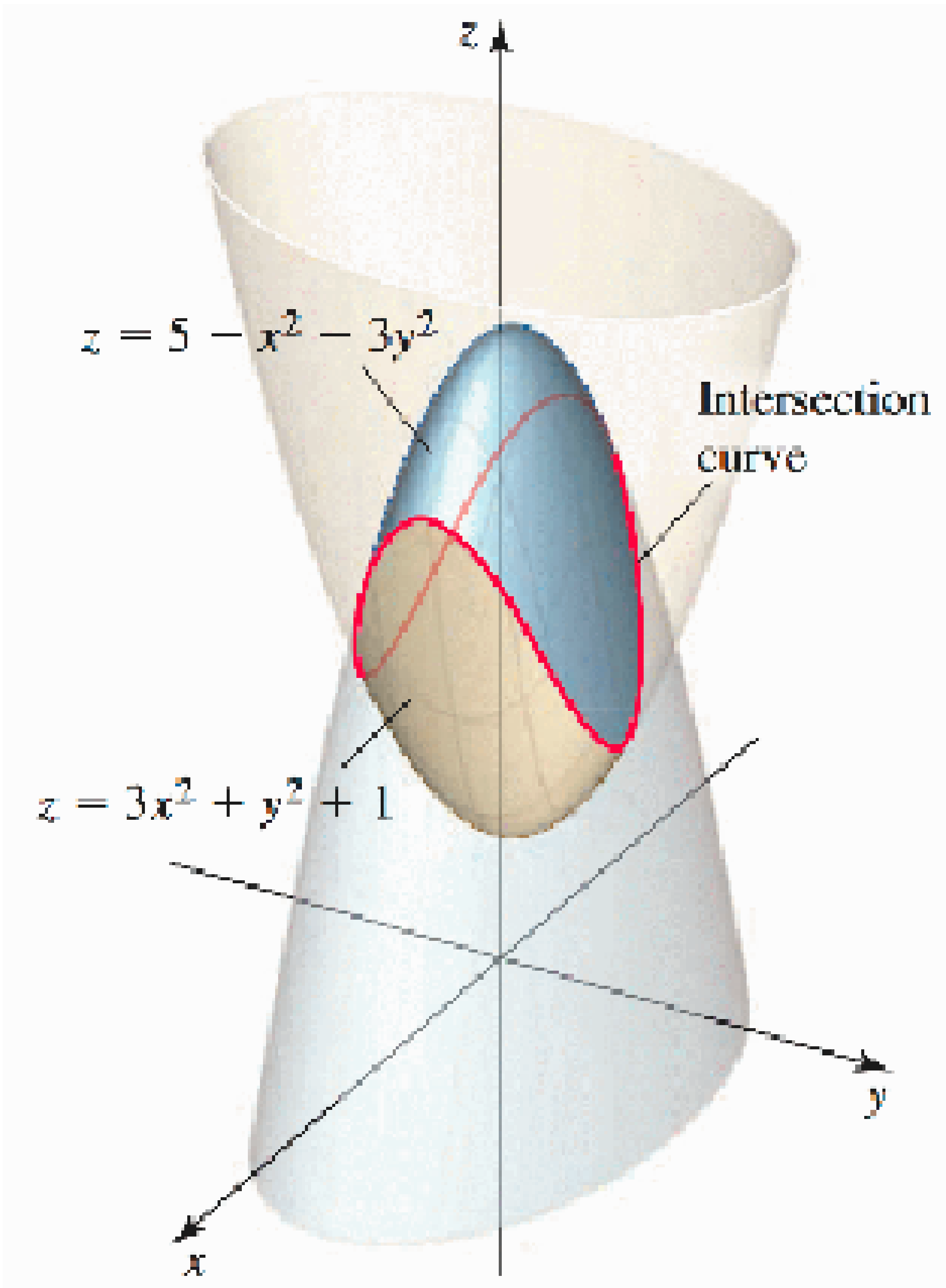
Remark: We can easily check our answer by verifying that R, S, and T all satisfy equation (8). Furthermore, we see that if we replace $\vec{n}$ by $c\vec{n}$ for any nonzero constant c, then $c\vec{n}$ will still be normal to P, which will result in seemingly different equations for P such as

$$-2 = 5x - 2y - 3z \tag{9}$$

when $c = -1$. In particular the order of the cross product in equation (3) and the order of the subtraction in equations (1) and (2) don't matter since the end result will at worst alter $\vec{n}$ by a negative sign.

Problem 3: Find a function $\vec{r}(t)$ that describes the curve $\mathcal{C}$ which is the intersection of the surfaces $z = 3x^2 + y^2 + 1$ and $z = 5 - x^2 - 3y^2$. Note that there is not a unique answer to this question since any curve possess infinitely many distinct paramterizations.

$$z = 3x^2 + y^2 + 1; z = 5 - x^2 - 3y^2$$



Solution: Writing $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we see that we only need to determine $x(t)$ and $y(t)$ since it will then be very easy to determine $z(t)$ using the defining equations of the surfaces in play. For the sake of readability we will write x, y, z instead of $x(t), y(t), z(t)$ for calculations in this problem. We see that

$$3x^2 + y^2 + 1 = z = 5 - x^2 - 3y^2 \to 4x^2 + 4y^2 = 4 \to x^2 + y^2 = 1. \tag{10}$$

It follows that $\langle x(t), y(t)\rangle$ traces out the unit circle, so we may set $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$. We now see that

$$z = 3x^2 + y^2 + 1 = 3\cos^2(t) + \sin^2(t) + 1 = 2\cos^2(t) + 2 = \cos(2t) + 3. \tag{11}$$

Putting everything together we see that we may take

$$\boxed{\vec{r}(t) = \langle \cos(t), \sin(t), \cos(2t) + 3\rangle, 0 \le t \le 2\pi}. \tag{12}$$

Problem 4: Determine whether the lines $\vec{r}(t) = \langle 1,3,2\rangle + t\langle 6,-7,1\rangle$ and $R(s) = \langle 10,6,14\rangle + s\langle 8,1,4\rangle$ are parallel or skew, and find their intersection(s) if any exist.

Solution: Let us first determine whether or not the lines paramterized by $\vec{r}(t)$ and $\vec{R}(s)$ are parallel since that requires the least computations. We see that the line paramterized by $\vec{r}(t)$ has the same direction as the vector $\langle 6,-7,1\rangle$ and the line parameterized by $\vec{R}(s)$ has the same direction as the vector $\langle 8,1,4\rangle$. It is clear that there is no constant c for which

$$\langle 6,-7,1\rangle = c\langle 8,1,4\rangle = \langle 8c, c, 4c\rangle \tag{13}$$

since we cannot simulataneously have $c = -7$ and $4c = 1$, so the lines in question are not parallel. Now let us search for the intersection(s) of the lines in questions while recalling that the lines will be skew if there are no intersections (since we have already shown that they are not parallel). To do this, we want to find all $t, s \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, which results in the following computations:

$$\underbrace{\langle 1,3,2\rangle + t\langle 6,-7,1\rangle}_{\vec{r}(t)} = \underbrace{\langle 10,6,14\rangle + s\langle 8,1,4\rangle}_{\vec{R}(s)} \tag{14}$$

$$\Leftrightarrow \langle 1+6t, 3-7t, 2+t\rangle = \langle 10+8s, 6+s, 14+4s\rangle \tag{15}$$

$$\Leftrightarrow \begin{array}{rcl} 1+6t &=& 10+8s \\ 3-7t &=& 6+s \\ 2+t &=& 14+4s \end{array} \tag{16}$$

$$\rightarrow s = -3-7t \tag{17}$$

$$\rightarrow 2+t = 14+4s = 14+4(-3-7t) = 2-28t \tag{18}$$

$$\rightarrow t = 0 \rightarrow s = -3. \tag{19}$$

However, since

$$1+6\cdot 0 \neq 10+8\cdot(-3), \tag{20}$$

we see that there are no $s, t \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, so the lines in question are skew.