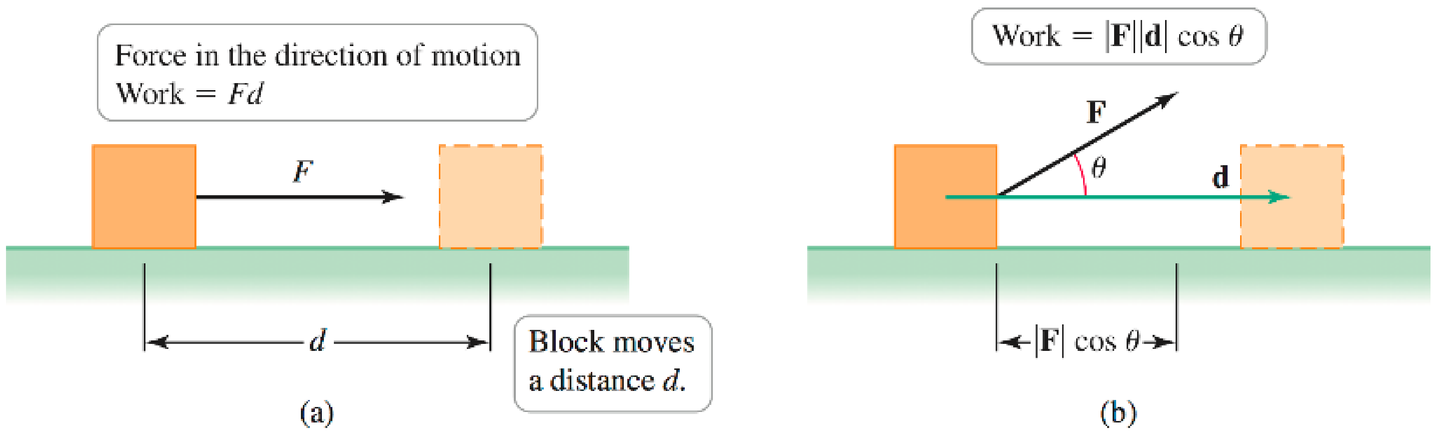
Problem 1: A suitcase is pulled 50ft along a horizontal sidewalk with a constant force of 30lb at an angle of 30° above the horizontal. How much work is done?

Solution: For this problem it suffices to use the formula for work that is shown in the diagram below.



The only thing that we need to be careful of is to remember that the standard unit of measure for work is Joules (J) which is given by $\mathrm{J} = \mathrm{kg}{\cdot}\mathrm{m}/s^2 = \mathrm{N}{\cdot}\mathrm{m}$, where N represents Newtons. To this end, we recall that 1lb≈4.4482N and 1ft≈0.3048m. It follows that the total amount of work done is given by

$$(1) \quad \text{Work} = 30\text{lb} \cdot 50\text{ft} \cdot \cos(30^\circ) \approx 30 \cdot 4.4482\text{N} \cdot 50 \cdot 0.3048\text{m} \cdot \frac{\sqrt{3}}{2} \approx \boxed{761.2506\text{J}}.$$

Problem 2: A constant force of $\vec{F} = \langle 2, 4, 1\rangle$N moves an object from $(0, 0, 1)$m to $(2, 4, 6)$m. How much work is done?

Solution: For this problem it helps to use the formula

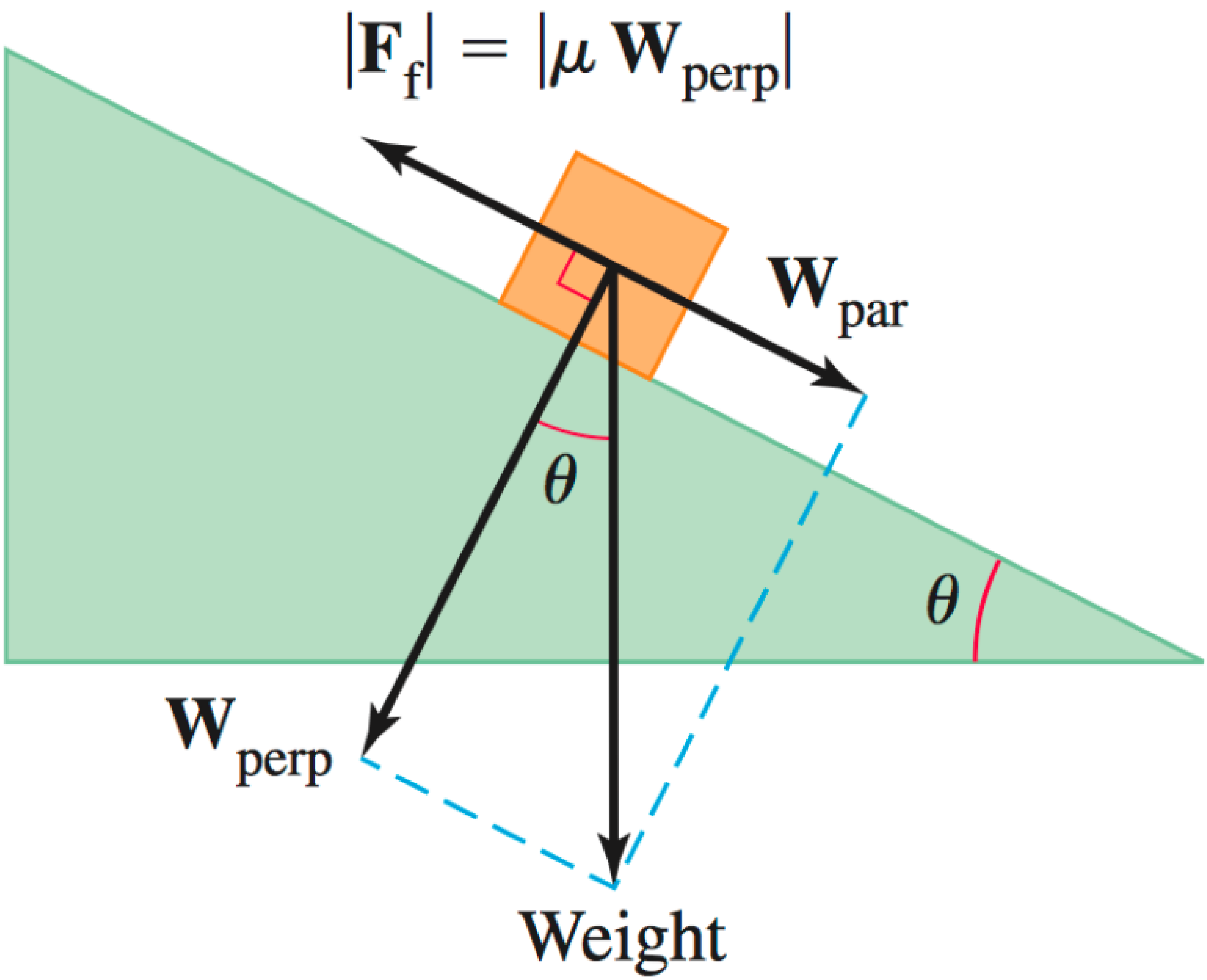
$$\text{Work} = |\vec{F}| \cdot |\vec{d}\,| \cos(\theta) = \vec{F} \cdot \vec{d}, \tag{2}$$

where $\vec{F}$ is a constant force that is applied to an object that moves in a straight line with a final displacement of $\vec{d}$. We now see that

$$\vec{d} = \langle 2, 4, 6\rangle m - \langle 0, 0, 1\rangle m = \langle 2, 4, 5\rangle m \tag{3}$$

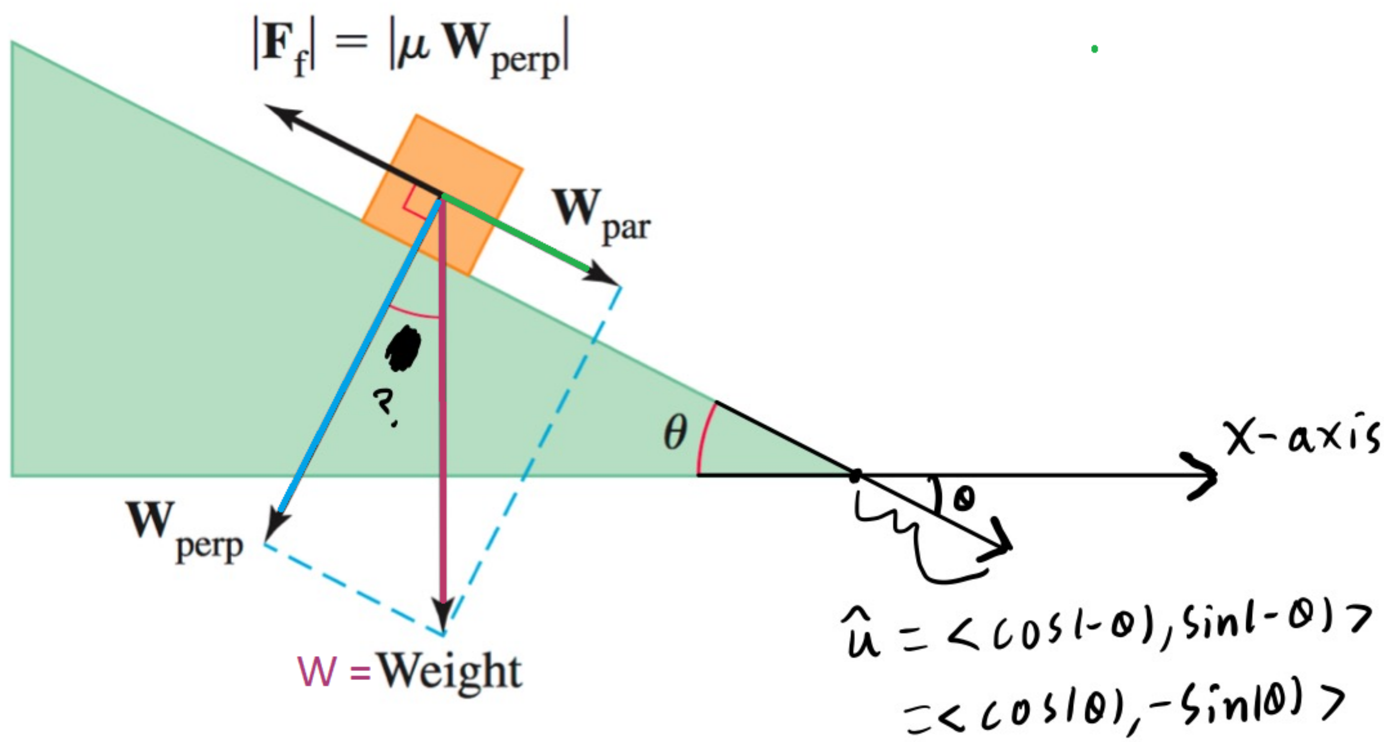
$$\to \text{Work} = \underbrace{\langle 2, 4, 1\rangle N}_{\vec{F}} \cdot \underbrace{\langle 2, 4, 5\rangle m}_{\vec{d}} = \boxed{25\text{J}}. \tag{4}$$

Problem 3: An object on an inclined plane does not slide provided the component of the object's weight parallel to the plane $|\mathbf{W}_{\text{par}}|$ is less than or equal to the magnitude of the opposing frictional force $|\mathbf{F}_f|$. The magnitude of the frictional force, in turn, is proportional to the component of the object's weight perpendicular to the plane $|\mathbf{W}_{\text{perp}}|$. The constant of proportionality is the coefficient of static friction $\mu > 0$. Suppose a 100lb block rests on a plane that is tilted at an angle of $\theta = 30°$ to the horizontal. What is the smallest possible value of μ?



We will present 2 solutions to this problem. The first solution is a direct approach but is computationally intensive. The second solution requires a little more ingenuity but is shorter. For the sake of generality, in both solutions we will solve the problem for a general angle θ and weight w and only plug in $\theta = 30°$ and $w = 100$ at the very end.

Solution 1: We see that $\mathbf{W} = \mathbf{W}_{\text{par}} + \mathbf{W}_{\text{perp}}$ is an decomposition of the force of gravity $\mathbf{W}$ into a sum of two orthogonal components. Since we know that $\mathbf{W} = \langle 0, -w\rangle$lb we only need to find $\mathbf{W}_{\text{par}}$ and it will then be easy to obtain $\mathbf{W}_{\text{perp}}$ through subtraction. To find $\mathbf{W}_{\text{par}}$ we calculate the orthogonal projection of $\mathbf{W}$ onto $\hat{u}$, the direction of the ramp as shown in the diagram below.



We now see that

$$\mathbf{W}_{\text{par}} = \text{Proj}_{\hat{u}}\mathbf{W} = \frac{\mathbf{W}\cdot\hat{u}}{|\hat{u}|^2}\hat{u} = (\mathbf{W}\cdot\hat{u})\hat{u} \tag{5}$$

$$= (\langle 0, -w\rangle \cdot \langle\cos(\theta), -\sin(\theta)\rangle)\langle\cos(\theta), -\sin(\theta)\rangle \tag{6}$$

$$= \langle w\sin(\theta)\cos(\theta), -w\sin(\theta)^2\rangle \tag{7}$$

$$\mathbf{W}_{\text{perp}} = \mathbf{W} - \mathbf{W}_{\text{par}} = \langle 0, -w\rangle - \langle w\sin(\theta)\cos(\theta), -w\sin(\theta)^2\rangle \tag{8}$$

$$= \langle -w\sin(\theta)\cos(\theta), w(-1+\sin^2\theta)\rangle = \langle -w\sin(\theta)\cos(\theta), -w\cos^2(\theta)\rangle \tag{9}$$

We now recall that we are searching for μ for which

$$(10) \qquad |\mathbf{W}_{\text{par}}| = |\mathbf{F}_{\text{f}}| = |\mu \mathbf{W}_{\text{perp}}| = \mu |\mathbf{W}_{\text{perp}}| \to \mu = \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|}$$

To this end, we see that[1]

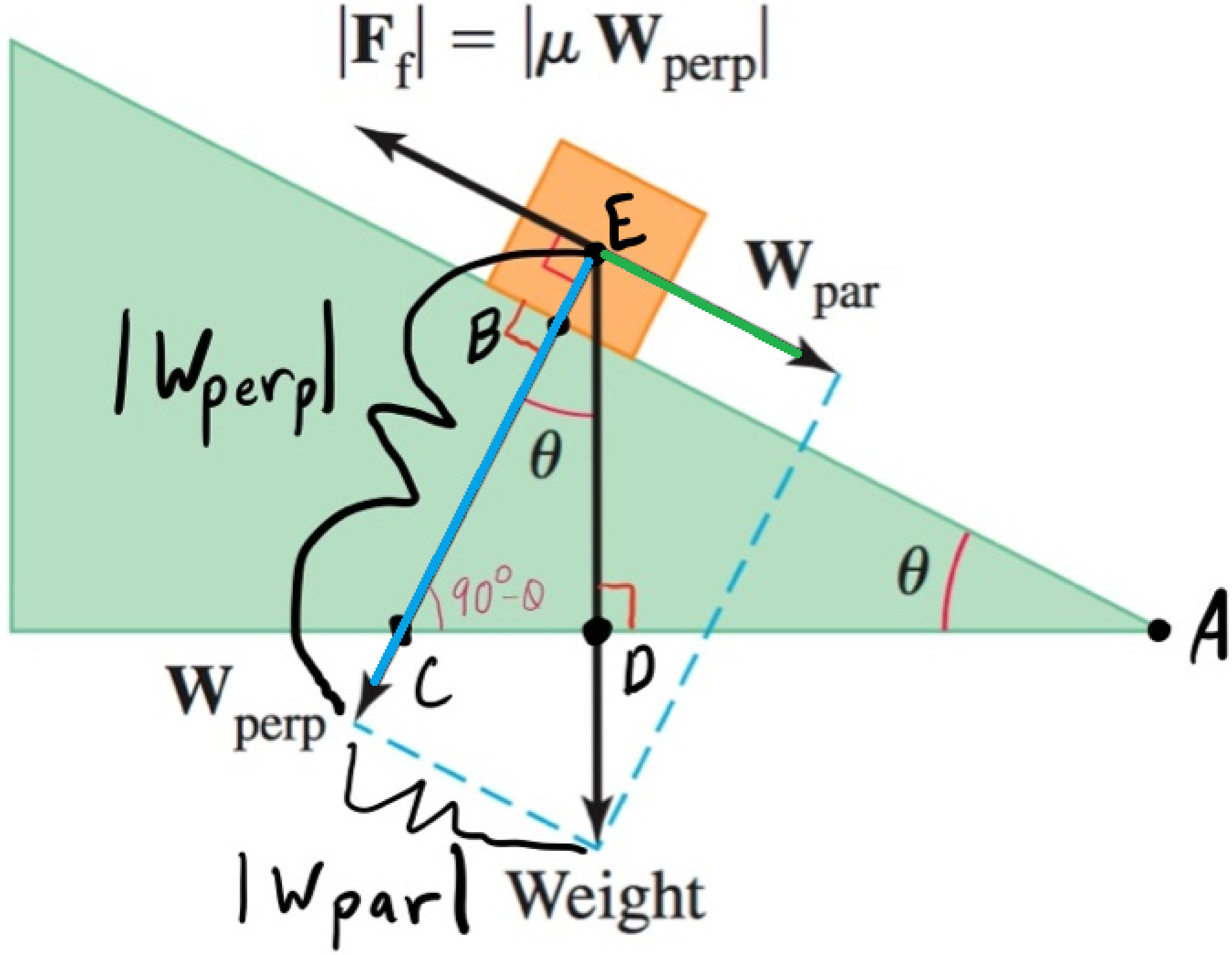
$$(11) \quad |\mathbf{W}_{\text{par}}| = \sqrt{(w\sin(\theta)\cos(\theta))^2 + (-w\sin(\theta)^2)^2} = w\sin(\theta)\sqrt{\cos^2(\theta)+\sin^2(\theta)} = w\sin(\theta), \text{ and}$$

$$(12) \quad |\mathbf{W}_{\text{perp}}| = \sqrt{(-w\sin(\theta)\cos(\theta))^2 + (-w\cos^2(\theta))^2} = w\cos(\theta)\sqrt{\sin^2(\theta)+\cos^2(\theta)} = w\cos(\theta), \text{ hence}$$

$$(13) \qquad \mu = \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} = \frac{w\sin(\theta)}{w\cos(\theta)} = \boxed{\tan(\theta)} = \tan(30^\circ) = \boxed{\frac{1}{\sqrt{3}}}.$$

Solution 2: First, let us verify that the two angles labeled with θ in the given diagram are indeed the same angle. We begin by labeling points on the original diagram as shown in the new diagram below in order to obtain the subsequent calculations.

[1] Recall that $\sin(\theta), \cos(\theta) \geq 0$ when $0 \leq \theta \leq 90^\circ$, so we don't need to write $|\sin(\theta)|$ or $|\cos(\theta)|$ in this case.



(14) $$\angle ACB = 90^\circ - \angle BAC = 90^\circ - \theta \text{ and}$$

(15) $$\angle CED = 90^\circ - \angle DCE = 90^\circ - \angle ACB = 90^\circ - (90^\circ - \theta) = \theta,$$

so the given diagram was indeed correctly labeled. We now recall that we are searching for μ for which

(16) $$|\mathbf{W}_{\text{par}}| = |\mathbf{F}_{\text{f}}| = |\mu \mathbf{W}_{\text{perp}}| = \mu|\mathbf{W}_{\text{perp}}| \to \mu = \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|}$$

After taking a look at our labeled diagram we see that

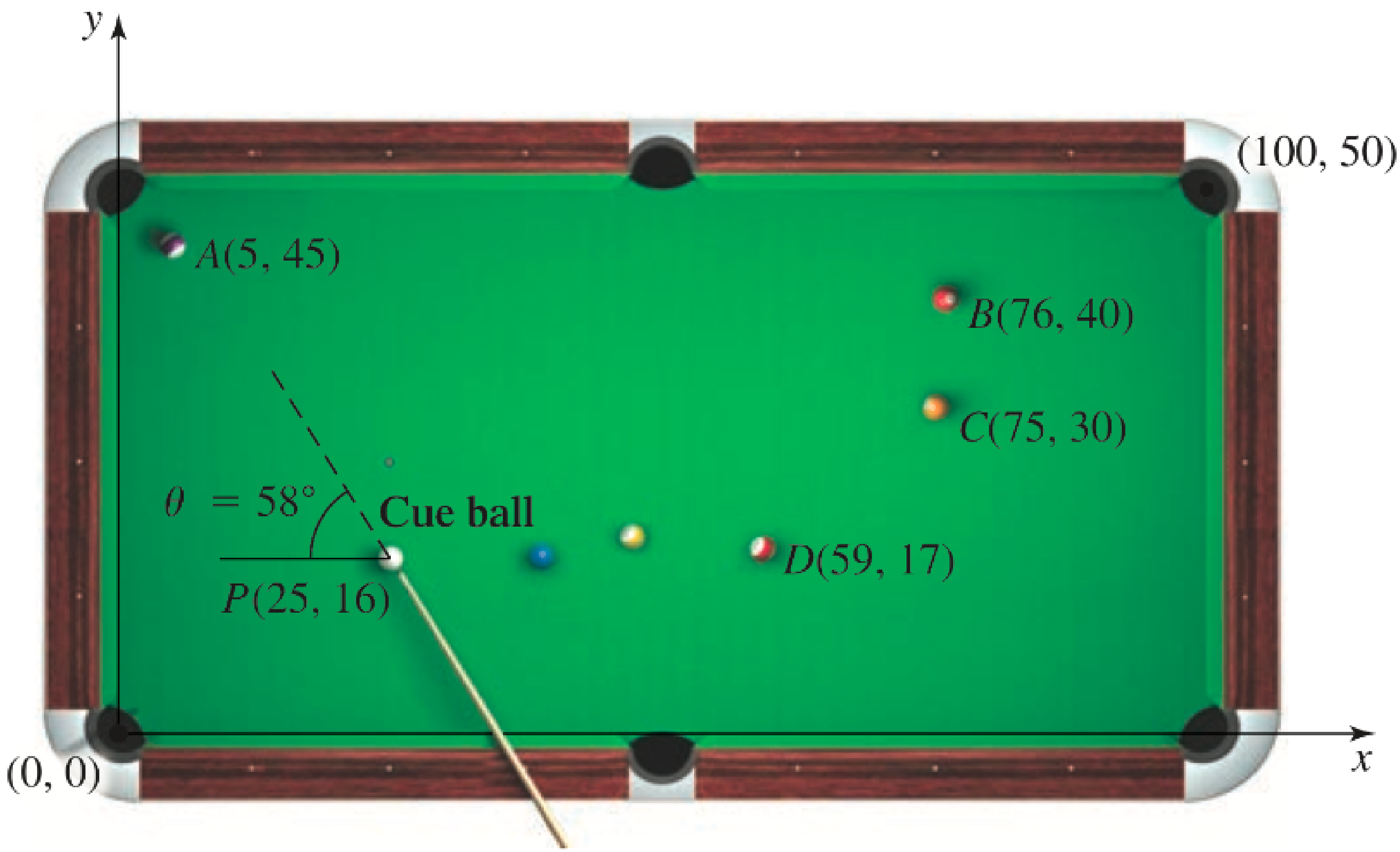
(17) $$\frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} = \boxed{\tan(\theta)} = \tan(30^\circ) = \boxed{\frac{1}{\sqrt{3}}}.$$

Problem 4: A cue ball in a billiards video game lies at P(25, 16). We assume that each ball has a diameter of 2.25 screen units, and pool balls are represented by the point at their center.

a. The cue ball is aimed at an angle of 58° above the negative x-axis toward a target ball at A(5, 45). Do the balls collide?

b. The cue ball is aimed at the point (50, 25) in an attempt to hit a target ball at B(76, 40). Do the balls collide?

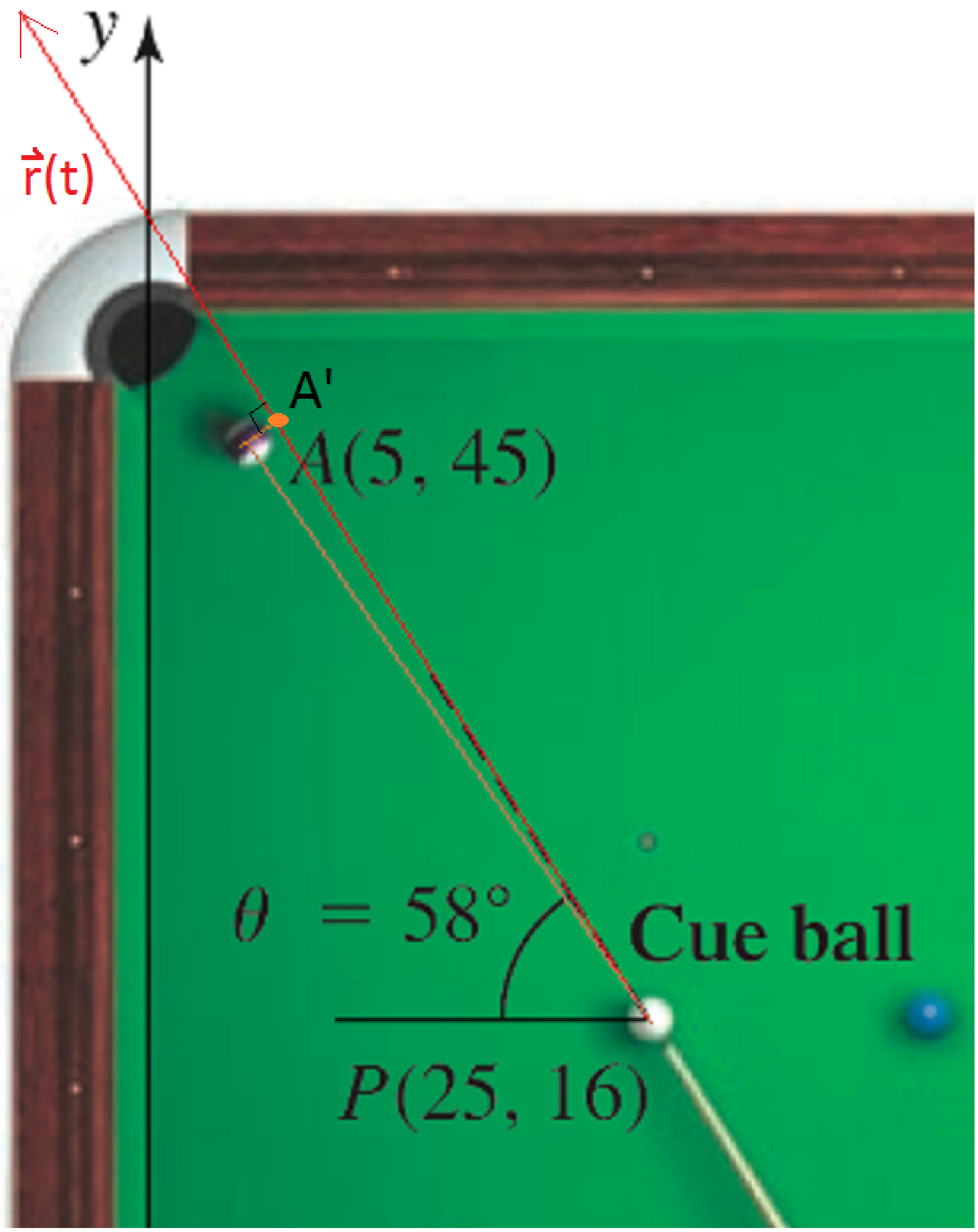
c. The cue ball is aimed at an angle θ above the x-axis in the general direction of a target ball at C(75, 30). What range of angles (for $0 \leq \theta \leq \frac{\pi}{2}$) will result in a collision? Express your answer in degrees.



Solution to a: Since the diameter of each ball is 2.25 units, the balls collide if at some point in time their centers are at most 2.25 units away from each other. Therefore we can solve this problem by calculating the distance between the point A and the straight line that is the trajectory of the cue ball. To this end we observe that the cue ball is aimed at an angle of $180° - 58° = 122°$ above the positive x-axis, so we can parametrize the trajectory of the cue ball (even though we really only need the direction of the trajectory) by

$$(18)\quad \vec{r}(t) = \langle 25, 16\rangle + t \underbrace{\langle \cos(122^\circ), \sin(122^\circ)\rangle}_{\hat{u}\text{, the direction of the trajectory}} = \langle 25 + t\cos(122^\circ), 16 + t\sin(122^\circ)\rangle.$$

Now let A' be the point on $\vec{r}(t)$ that is closest to A, which happens to be the orthogonal projection of A onto $\vec{r}(t)$ as shown in the diagram below.



We have now reduced to problem down to whether or not $|\overrightarrow{A'A}|$ is larger than 2.25 or not. Since $\overrightarrow{A'A} = \overrightarrow{PA} - \overrightarrow{PA'}$, we first observe that

$$\overrightarrow{PA} = \underbrace{\langle 5, 45\rangle}_{\text{A}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle -20, 29\rangle, \tag{19}$$

and we will now proceed to find $\overrightarrow{PA'}$. We see that $\overrightarrow{PA'}$ is the orthogonal projection of $\overrightarrow{PA}$ onto $\vec{r}(t)$, but $\hat{u}$ points in the same direction as $\vec{r}(t)$, so $\overrightarrow{PA'}$ is also the orthogonal projection of $\overrightarrow{PA}$ onto $\hat{u}$, hence

$$\overrightarrow{PA'} = (\overrightarrow{PA} \cdot \hat{u})\hat{u} \tag{20}$$

$$= \Big(\langle -20, 29\rangle \cdot \langle \cos(122°), \sin(122°)\rangle\Big)\langle \cos(122°), \sin(122°)\rangle \tag{21}$$

$$\approx \langle -18.65, 29.84\rangle. \tag{22}$$

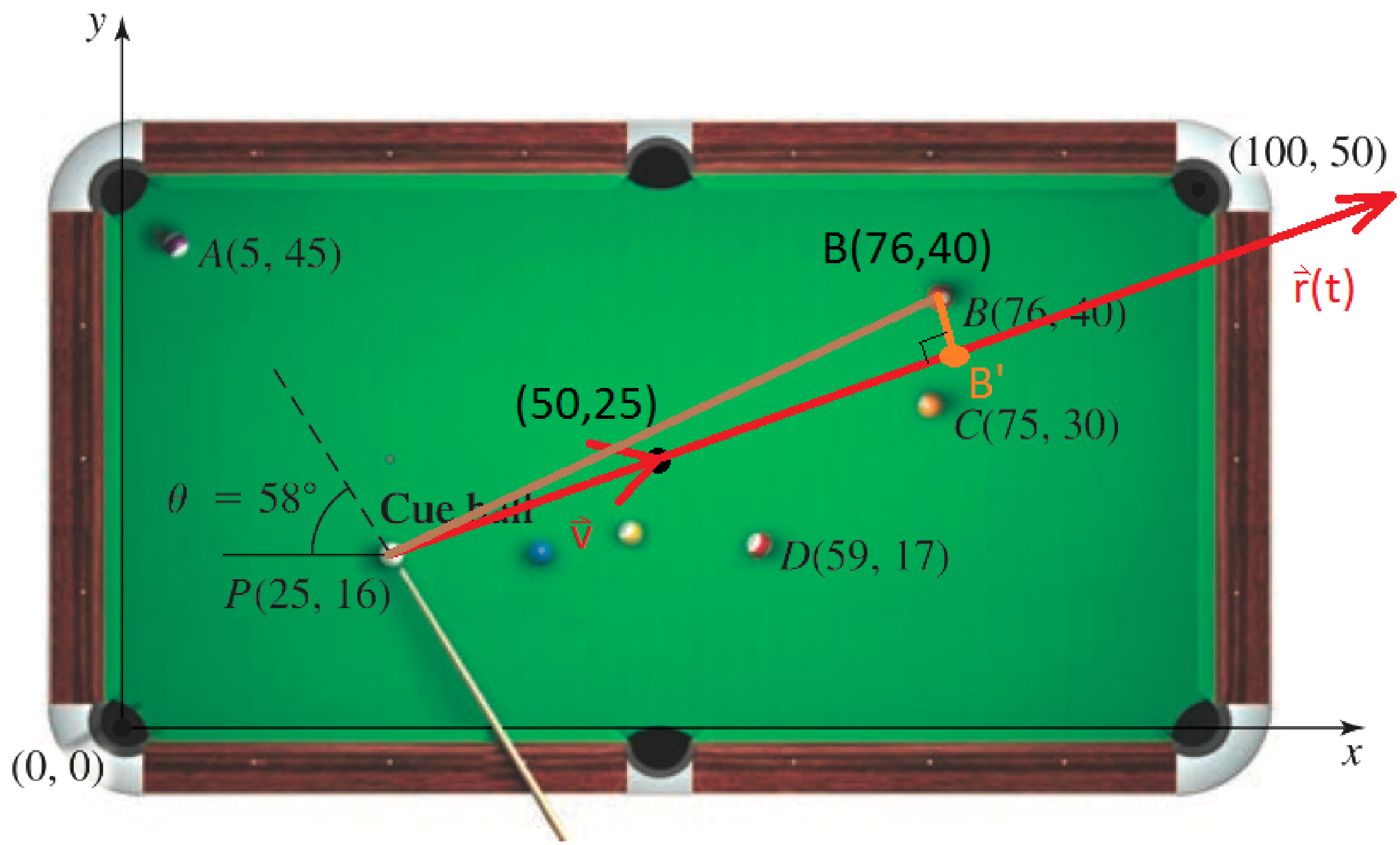
Putting everything together we see that

$$|\overrightarrow{A'A}| = |\overrightarrow{PA} - \overrightarrow{PA'}| \approx |\langle -20, 29\rangle - \langle -18.65, 29.84\rangle| \tag{23}$$

$$= |\langle -1.35, -0.84\rangle| = 1.59 < 2.25, \tag{24}$$

so the balls do collide.

Solution to b: We use the same strategy that we used in part a. The only difference is that we will use slightly different computations to obtain a parametrization of (or more importantly, the direction of) the path of the cue ball since we were given a point on its trajectory rather than the angle that the trajectory makes with the x-axis.



We see that

$$\vec{r}(t) = \langle 25, 16\rangle + t\ \underbrace{(\langle 50, 25\rangle - \langle 25, 16\rangle)}_{\text{Points in the direction of } \vec{r}(t)} \tag{25}$$

$$= \langle 25, 16\rangle + t\,\underbrace{\langle 25, 9\rangle}_{\vec{v}} = \langle 25 + 25t, 16 + 9t\rangle \tag{26}$$

Since $\overrightarrow{BB'} = \overrightarrow{PB} - \overrightarrow{PB'}$, we first observe that

$$\overrightarrow{PB} = \underbrace{\langle 76, 40\rangle}_{\text{B}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle 51, 24\rangle, \tag{27}$$

and we will now proceed to find $\overrightarrow{PB'}$. Since $\overrightarrow{PB'}$ is the orthogonal projection of $\overrightarrow{PB}$ onto $\vec{r}(t)$ we see as in part a that $\overrightarrow{PB'}$ is also the orthogonal projection of $\overrightarrow{PB}$ onto $\vec{v}$, so

$$\overrightarrow{PB'} = \frac{\overrightarrow{PB} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\langle 51, 24\rangle \cdot \langle 25, 9\rangle}{|\langle 25, 9\rangle|^2}\langle 25, 9\rangle \tag{28}$$

$$= \langle \frac{37275}{706}, \frac{13419}{706} \rangle \approx \langle 52.80, 19.01 \rangle \tag{29}$$

Putting everything together we see that

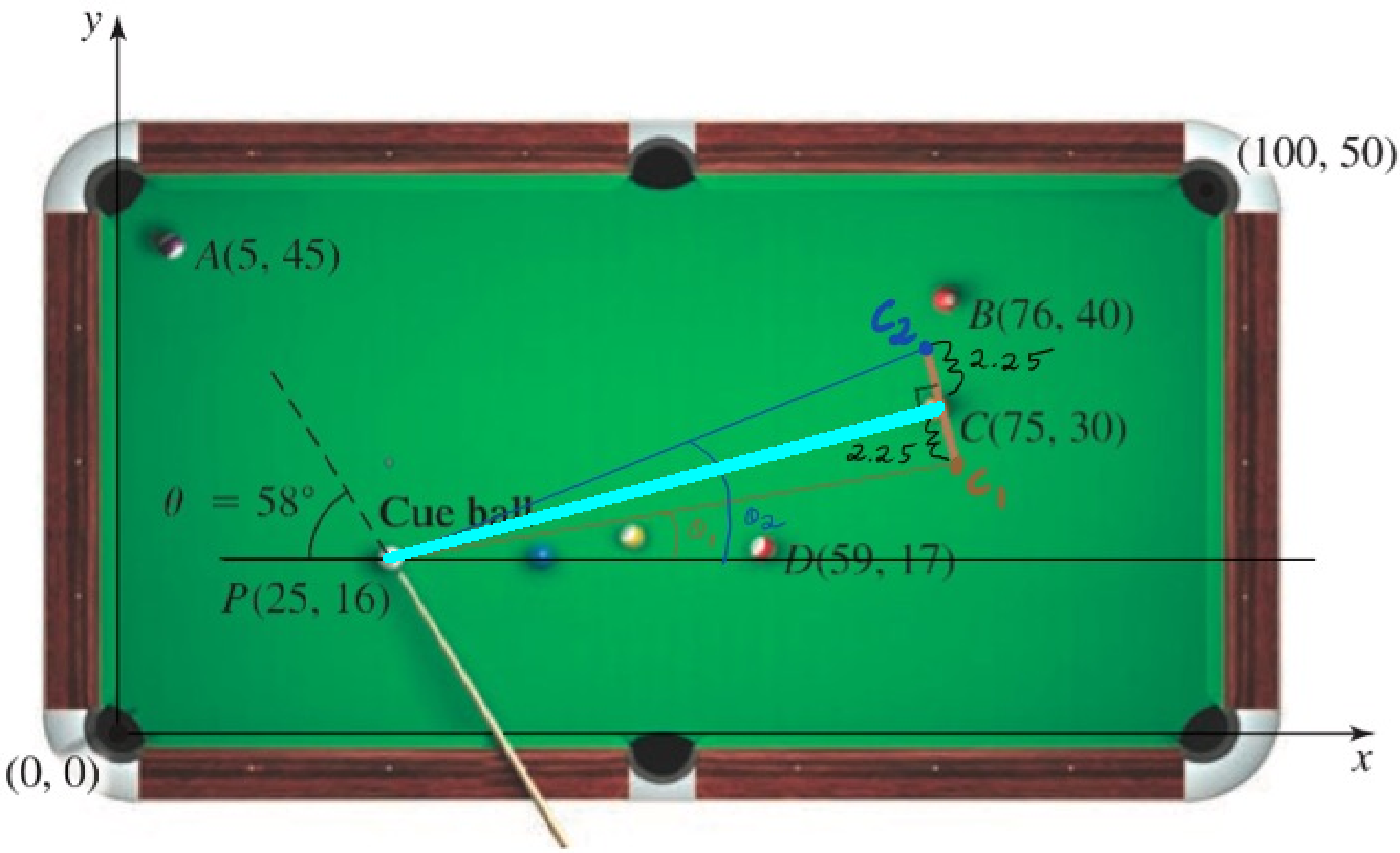
$$|\overrightarrow{BB'}| = |\overrightarrow{PB} - \overrightarrow{PB'}| \approx |\langle 51, 24 \rangle - \langle 52.80, 19.01 \rangle| \tag{30}$$

$$|\langle -1.80, 4.99 \rangle| \approx 5.30 > 2.25, \tag{31}$$

so the balls do not collide.

Remark: While we did not use the parametrizations $\vec{r}(t)$ in either of parts a or b to calculate the relevant orthogonal projections, it is worth noting that an alternative solution to these problems is to use single variable calculus to find the minimum value of the function $d_A(t) = d((5, 45), \vec{r}(t))$ for part a and the function $d_B(t) = d((76, 40), \vec{r}(t))$ for part b.

Solution to c: Let the points C_1 and C_2 be such that $|\overrightarrow{CC_1}| = |\overrightarrow{CC_2}| = 2.25$ and each of $\overrightarrow{CC_1}$ and $\overrightarrow{CC_2}$ are orthogonal to $\overrightarrow{PC}$. $\overrightarrow{PC_1}$ and $\overrightarrow{PC_2}$ represent the trajectories in which the cue ball just barely touches the ball at C, so we want to determine the angles θ_1 and θ_2 as shown in the diagram below.



To this end we begin by observing that

(32) $$\overrightarrow{PC} = \underbrace{\langle 75, 30\rangle}_{\text{C}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle 50, 14\rangle.$$

Recalling that for a given vector $\vec{w} := \langle x, y\rangle$ the vectors $\langle -y, x\rangle$ and $\langle y, -x\rangle$ are orthogonal to $\vec{w}$, we see that

(33) $$\overrightarrow{PC_1} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle 14, -50\rangle}{|\langle 14, -50\rangle|} \approx \langle 50.61, 11.83\rangle, \text{ and}$$

(34) $$\overrightarrow{PC_2} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle -14, 50\rangle}{|\langle -14, 50\rangle|} \approx \langle 49.39, 16.17\rangle.$$

We now see that

(35) $$\theta_1 = \tan^{-1}\left(\frac{11.83}{50.61}\right) \approx 13.16°, \text{ and} \theta_2 = \tan^{-1}\left(\frac{16.17}{49.39}\right) \approx 18.13°,$$

so the balls will collide if $13.16° \leq \theta \leq 18.13°$

Remark: An alternative approach to solving this problem is to first find a parametrization $\vec{r_\theta}(t)$ of the trajectory of the cue ball when it makes an angle of $0 \le \theta \le \frac{\pi}{2}$ with the positive x-axis using the techniques from part a. Then for each such θ we use single variable calculus find $m(\theta)$, the minimum value of $d_{C,\theta}(t) = d((75, 30), \vec{r_\theta}(t))$ as t varies (and θ is still fixed). To finish the problem we find the values of θ for which $m(\theta) = 2.25$.