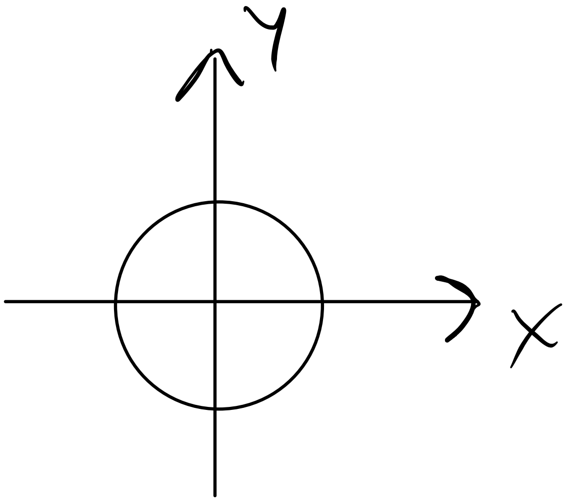
# Parametric Equations

10:20–11:15, Pg 1–9
11:30–12:25, Pg 10–14
12:40–1:35, Pg 15–19

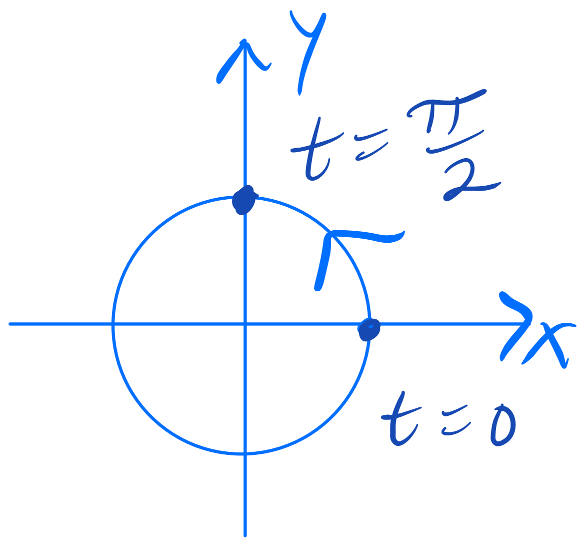
Parametric Eqs let us study motion.

$x^2+y^2=1$

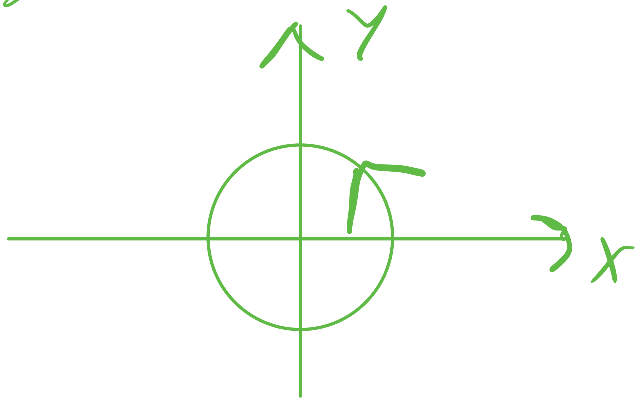



$\vec{r}(t)=\langle \cos(t), \sin(t)\rangle$

$0 \le t \le 2\pi$

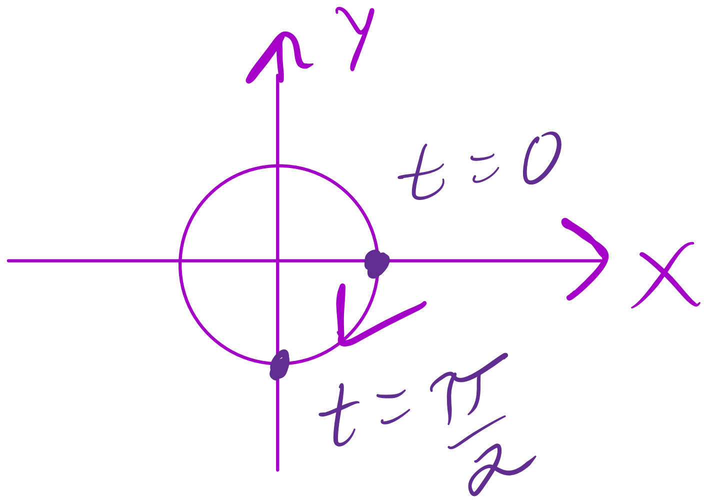



$\vec{r}(t)=\langle \cos(2t), \sin(2t)\rangle$, $0 \le t \le \pi$




Traverses the circle twice as fast as $\vec{r}(t)$.

$$\vec{r}(t) = \langle \cos(-t), \sin(-t) \rangle$$
$$= \langle \cos(t), -\sin(t) \rangle,$$
$$0 \le t \le 2\pi$$




$$(\cos(t))^2 + (-\sin(t))^2 = 1$$

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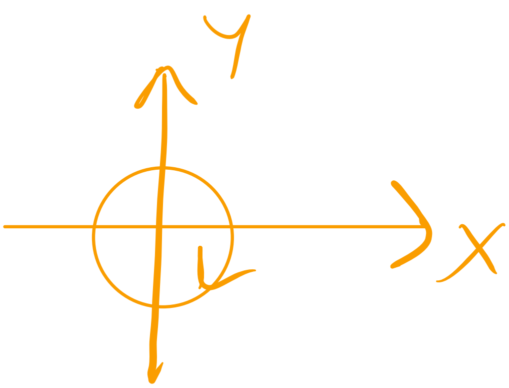
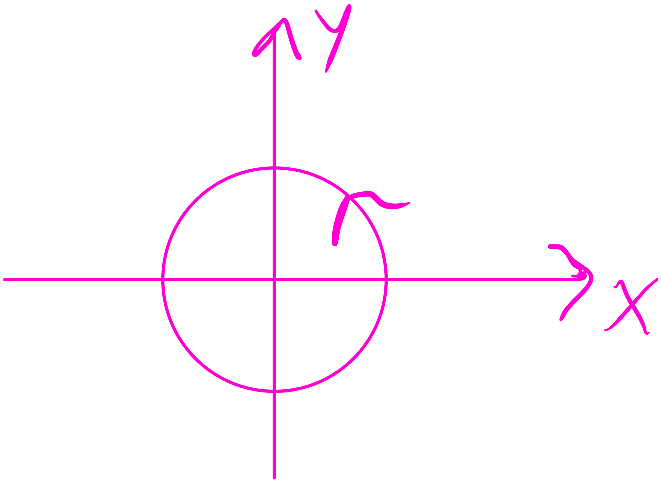
$$\vec{r}(t) = \langle \cos(-2t), \sin(-2t) \rangle$$
$$= \langle \cos(2t), -\sin(2t) \rangle$$
$$0 \le t \le 2\pi$$




Traverses the circle twice as fast as $\vec{r}(t)$.

$\vec{r}(t) = \langle \cos(t^2), \sin(t^2) \rangle$

$0 \le t \le \sqrt{2\pi}$




$\vec{r}(t)$ traverses the circle with non-constant speed.

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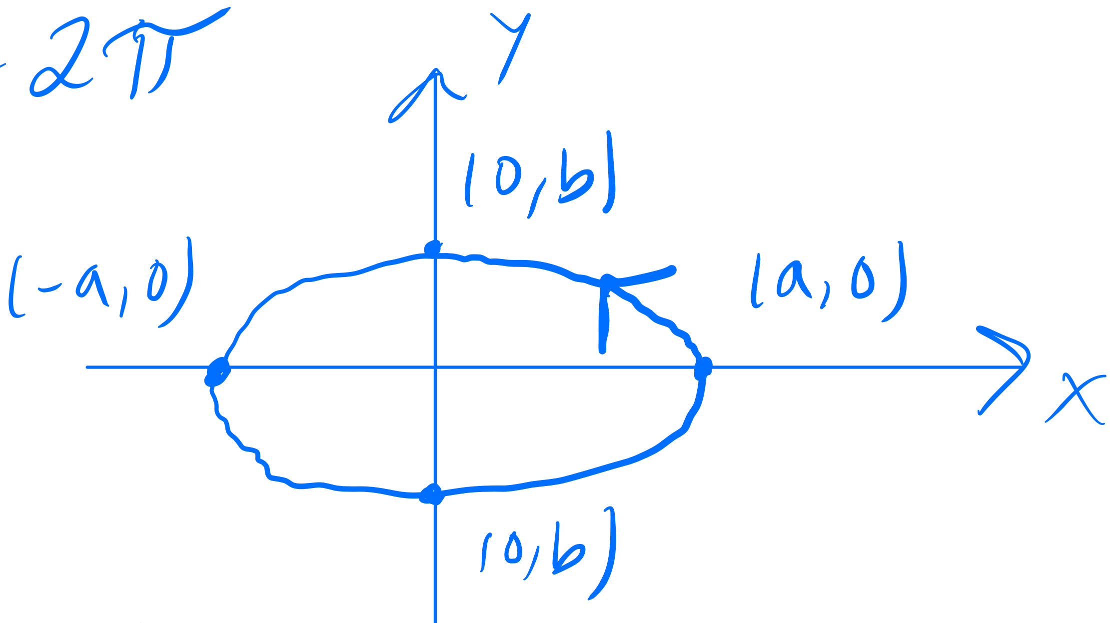
$\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$

$0 \le t \le 2\pi$




$\vec{r}(t)$ is the standard parametrization for an ellipse.

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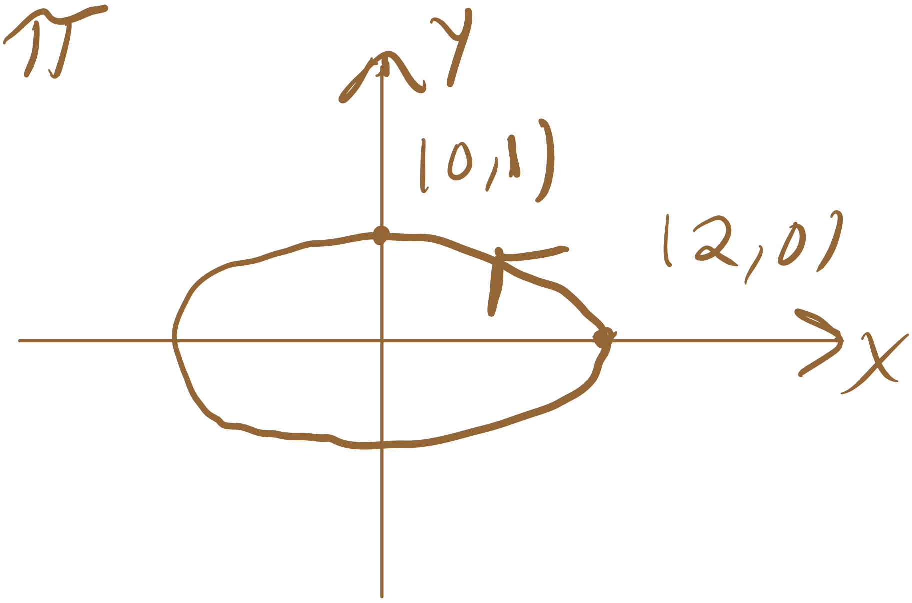
$\vec{r}(t) = \langle 2\cos(t), \sin(t) \rangle$

$0 \le t \le 4\pi$




$\vec{r}(t)$ traverses the ellipse twice.

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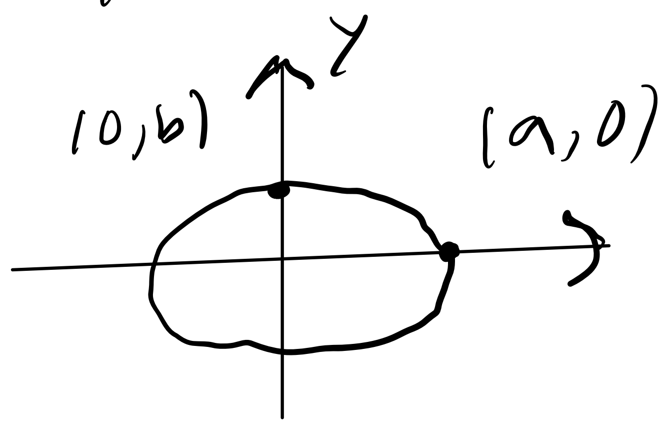
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is the standard equation of an ellipse.



# Parametrization of Lines

There are 3 ways to describe a line.

1) Give 2 points on the line.

2) The intersection of 2 (non-parallel) planes

3) Give a point on the line and the direction of the line.

Most commonly used.

c.f. Pool Problem Parts a and b

# Continuity

The vector-valued func.

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$,

$a \le t \le b$ is $\boxed{\text{continuous}}$

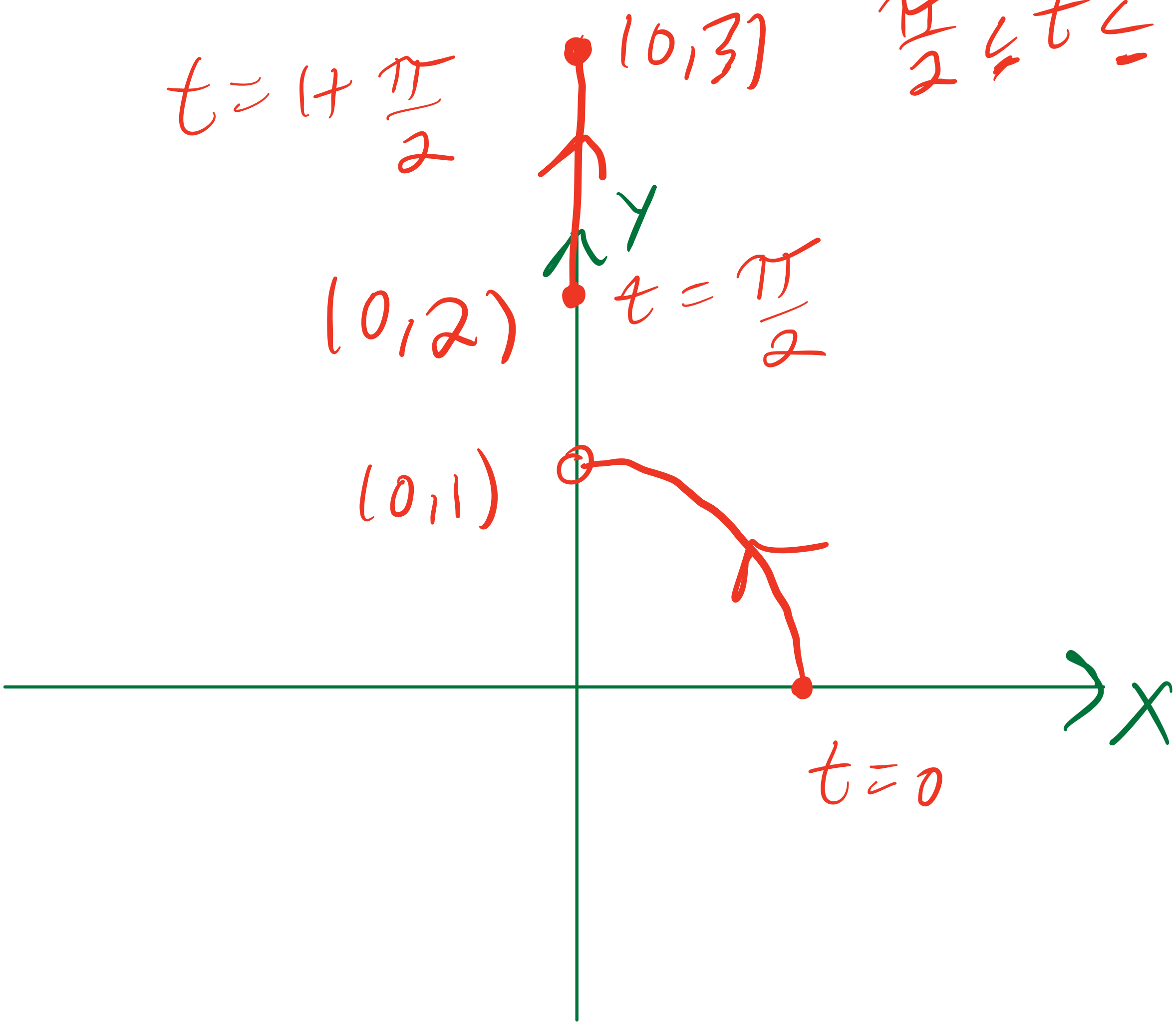
if for every $t_0 \in [a,b]$,

we have

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

Consider the discontinuous

$$\vec{r}(t)=\begin{cases}\langle \cos(t), \sin(t)\rangle & \text{if } 0\le t<\frac{\pi}{2}\\ \langle 0, 2+t-\frac{\pi}{2}\rangle & \text{if } \frac{\pi}{2}\le t\le 1+\frac{\pi}{2}\end{cases}$$




$$\lim_{t\to\frac{\pi}{2}^-}\vec{r}(t)=\langle 0,1\rangle\neq\langle 0,2\rangle=\vec{r}\left(\tfrac{\pi}{2}\right)$$

$\Rightarrow$ $\vec{r}(t)$ is discontinuous.

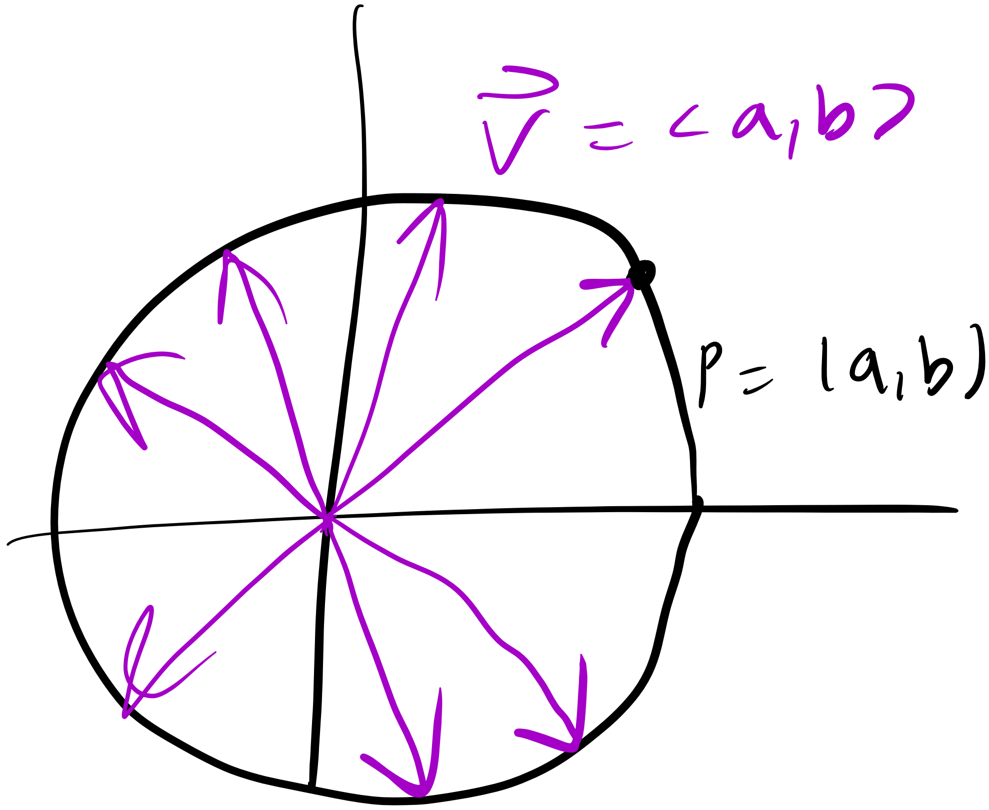
$$\lim_{t \to \frac{\pi}{2}^+} \vec{r}(t) = \langle 0, 2 \rangle = \vec{r}\left(\frac{\pi}{2}\right)$$

$$\lim_{t \to \frac{\pi}{2}^-} \vec{r}(t) \neq \lim_{t \to \frac{\pi}{2}^+} \vec{r}(t) \rightarrow$$

$$\lim_{t \to \frac{\pi}{2}} \vec{r}(t)$$ DNE! $\rightarrow$ $\vec{r}(t)$

is not continuous at $\frac{\pi}{2}$.

- $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$ is continuous if and only if each of $x(t), y(t),$ and $z(t)$ are continuous on $[a, b]$.

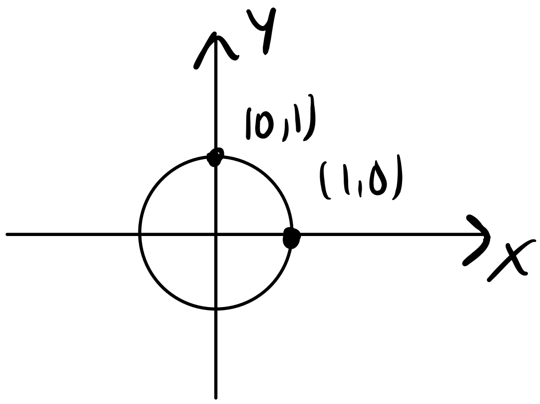



$$\vec{r}(t) = (\cos(t), \sin(t))$$

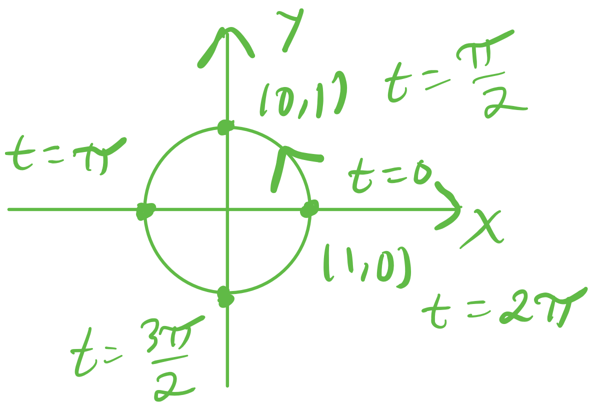
# Parametric Equations

Parametric equations are useful for studying motion.

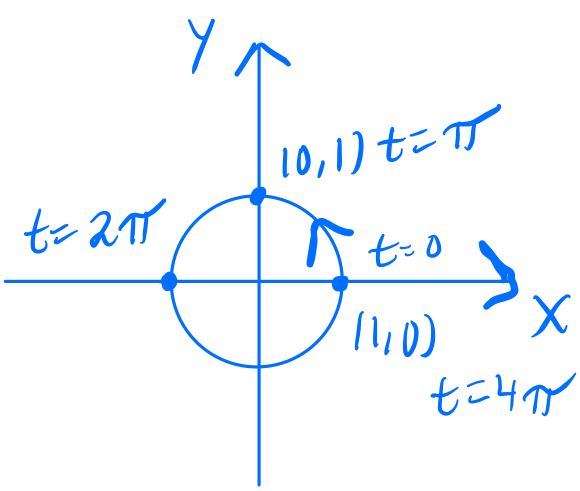
$$x^2+y^2=1$$




$\vec{r}(t)=\langle \cos(t), \sin(t)\rangle$, $0\le t\le 2\pi$

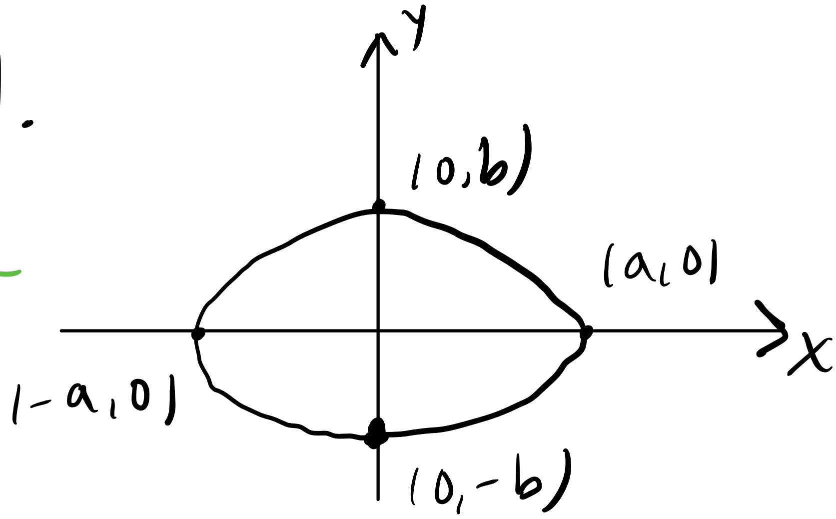



$\vec{r}(t)=\langle \cos(\frac{1}{2}t), \sin(\frac{1}{2}t)\rangle$, $0\le t\le 4\pi$




$\vec{r}(t)$ traverses the circle half as quickly as $\vec{r}(t)$.

The standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.




Circle is $x^2+y^2=r^2$

$$\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$$

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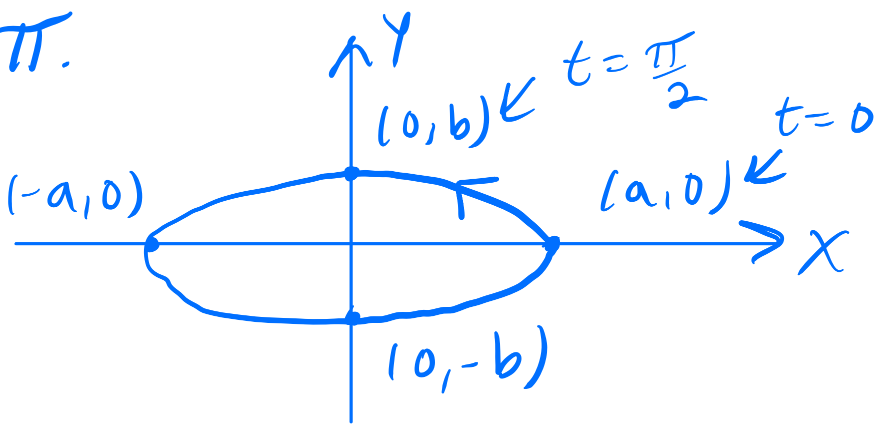
The standard parametrization of an ellipse is $\vec{r}(t)=\langle a\cos(t), b\sin(t)\rangle$, $0\le t\le 2\pi$.

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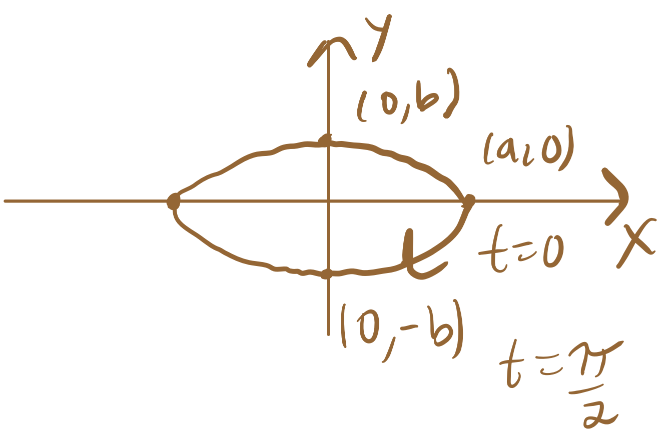
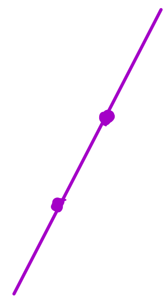
$\vec{r}(t)=\langle a\cos(-t), b\sin(-t)\rangle$, $0\le t\le 2\pi$




$\vec{r}(t)$ traverses the ellipse in the $\boxed{\text{clockwise}}$ direction.

# Parametrizations of Lines

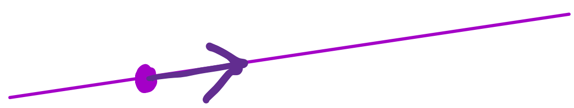
The three ways to describe a line are



1) Give 2 points on the line.

2) as the intersection of 2 (non-parallel) planes.

3) Give 1 point on the line <u>and</u> the "direction of the line."



Most commonly used for us.
C.f. Pool Problem Parts a and b.

Example: Find the line through the point $P=(1,2)$ which points in the direction of $\vec{v}=\langle 2,1\rangle$.

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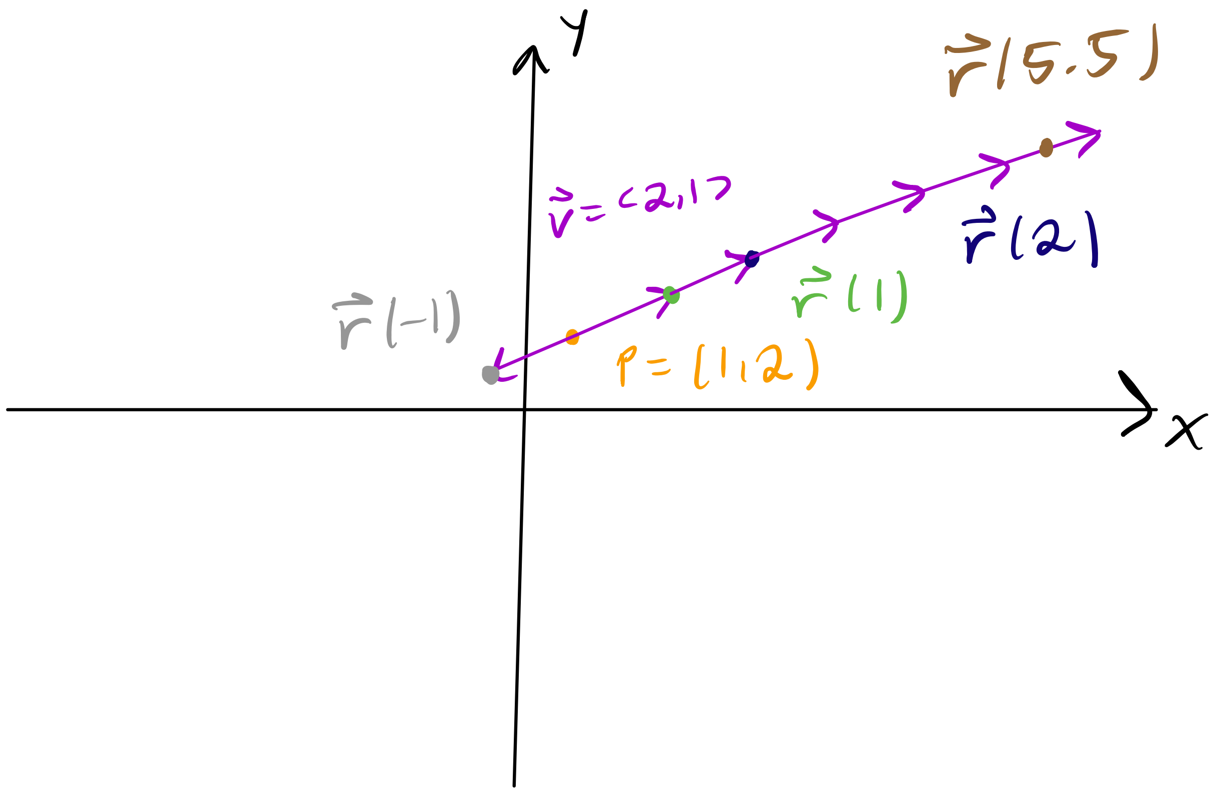
Solution: $\vec{r}(t) = P + t\vec{v}, \quad t \in \mathbb{R}$

$= \langle 1,2\rangle + t\langle 2,1\rangle$

$$\boxed{= \langle 1+2t, 2+t\rangle, \quad t \in \mathbb{R}}$$




Every line in $\mathbb{R}^3$ $\boxed{\text{can}}$ be parametrized as $\vec{r}(t) = \langle a_1 + c_1 t, a_2 + c_2 t, a_3 + c_3 t\rangle, \quad t \in \mathbb{R}$ for some $a_1, a_2, a_3, c_1, c_2, c_3 \in \mathbb{R}$.

$\vec{r}(t) = \langle t^2, 2t^2, 0 \rangle$, $2 \le t \le 5$ is a parametrization of a line segment.

## Continuity of Vector-valued functions

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$ is continuous if for every $t_0 \in [a,b]$,

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

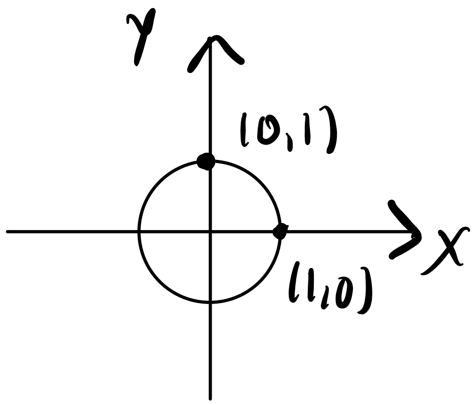
(In particular, the lim exists)

- $\vec{r}(t)$ is continuous if and only if $x(t)$, $y(t)$, and $z(t)$ are all continuous on $[a,b]$.

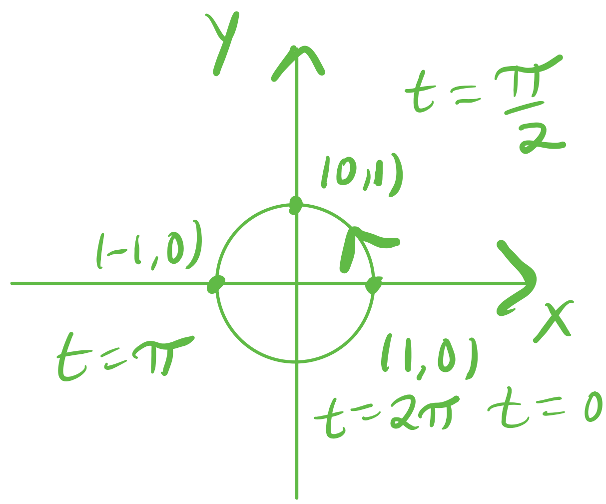
# Parametric Equations

Parametric equations are used to study motion.

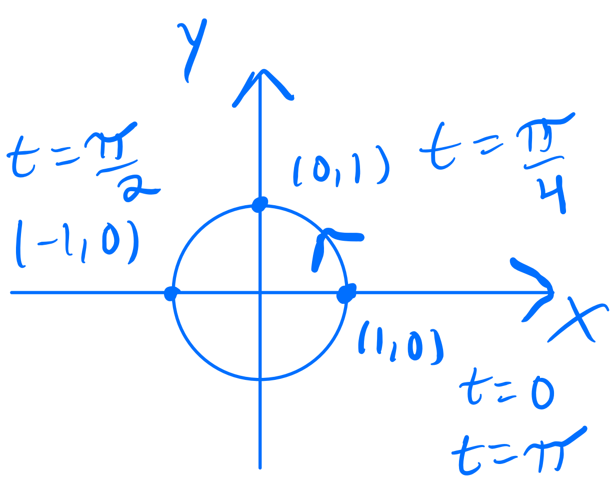
$x^2+y^2=1$




$\vec{r}(t)=\langle\cos(t),\sin(t)\rangle,$
$0\le t\le 2\pi$




$\vec{r}(t)=\langle\cos(2t),\sin(2t)\rangle,\ 0\le t\le\pi$



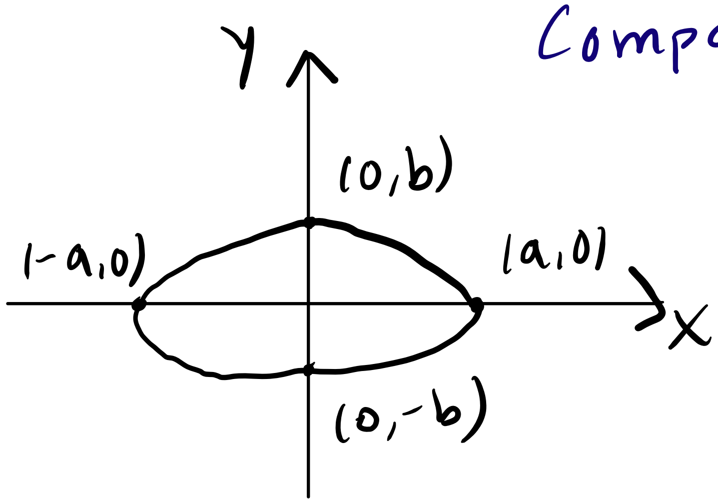

$\vec{r}(t)$ traverses the circle twice as fast as $\vec{r}(t)$.

The standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Compare to $x^2+y^2=r^2$

$\Updownarrow$

$\frac{x^2}{r^2}+\frac{y^2}{r^2}=1$

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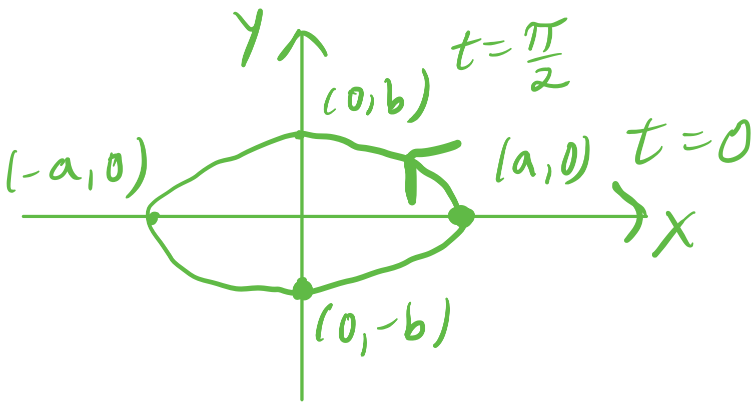
The standard parametrization of an ellipse is $\vec{r}(t)=\langle a\cos(t), b\sin(t)\rangle$, $0\le t\le 2\pi$.

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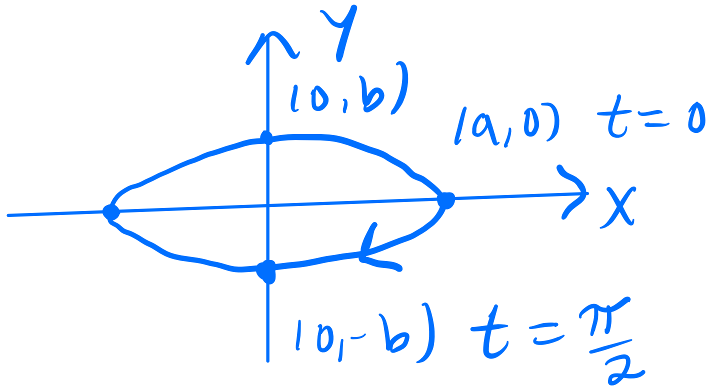


$\vec{r}(t)$ traverses the ellipse in the clockwise direction.

# Parametrization of Lines

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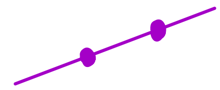
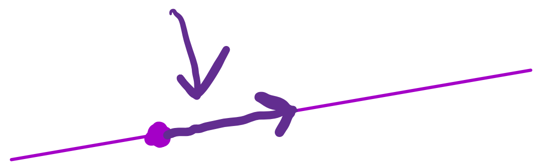
1) Give 2 points on the line. 

2) The intersection of 2 (non-parallel) planes.

3) Give 1 point on the line and the "direction" of the line.



Most used for us.

C.f. Pool Problem Parts a and b

Example: Find a parametrization for the line through $P = (1,2)$, which points in the direction of $\vec{v} = \langle 2,1 \rangle$.

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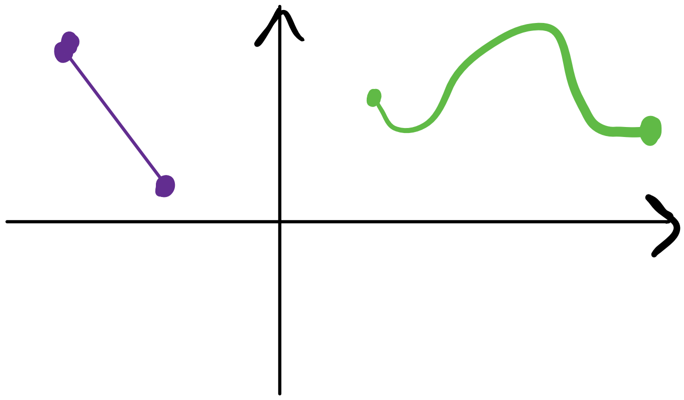
Solution: $\vec{r}(t) = P + t\vec{v},\ t \in \mathbb{R}$

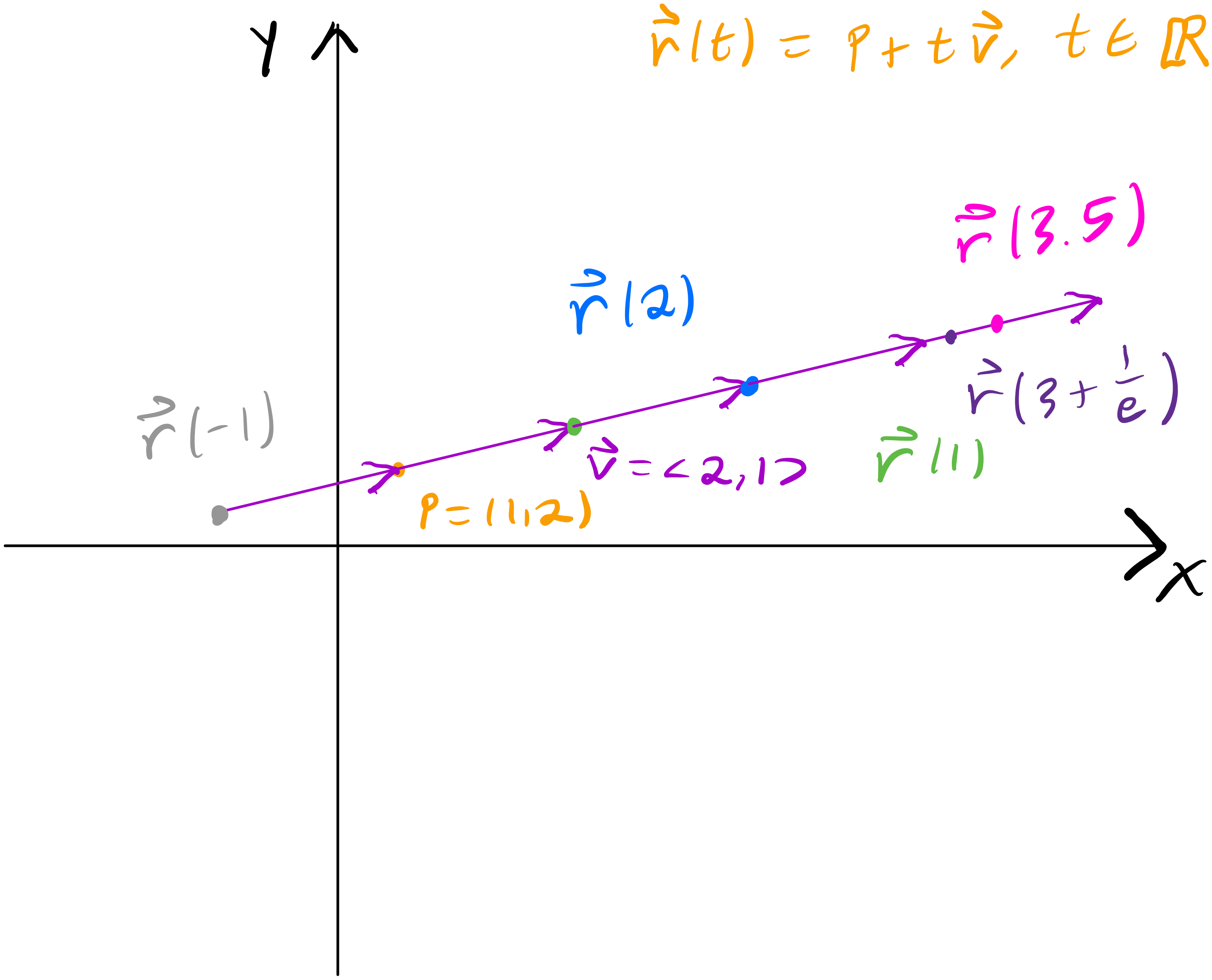
$= \langle 1,2 \rangle + t\langle 2,1 \rangle = \langle 1,2 \rangle + \langle 2t,t \rangle$

$\boxed{= \langle 1+2t,\ 2+t \rangle,\ t \in \mathbb{R}}$

density $= d(x,y) = x^2 - y$



$\vec{r}(t) = p + t\vec{v}, \; t \in \mathbb{R}$



# Continuity of Vector-valued functions

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is continuous if and only if

for any $t_0 \in [a,b]$, we have

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

f f

→ Calc I continuity

$\left( \lim_{t \to t_0} \vec{r}(t) \text{ must exist} \right)$

- $\vec{r}(t)$ is continuous if and only if $x(t)$, $y(t)$, and $z(t)$ are each continuous.