Problem 1: Consider

$$\int_{\mathcal{C}} (x^2 + y^2) ds,$$

where $\mathcal{C}$ is the line segment from $(0, 0)$ to $(5, 5)$.

(1) Find a parametric description for $\mathcal{C}$ in the form $\vec{r}(t) = \langle x(t), y(t) \rangle$. (*Remember to state the domain of the parameter.*)

(2) Evaluate $|\vec{r'}(t)|$.

(3) Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.

Solution to (1): We recall that

$$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}), 0 \leq t \leq 1 \tag{1}$$

is one way in which to parameterize the line segment that starts at the point P and ends at the point Q. In this particular problem, we see that

$$\vec{r}(t) = \langle 0, 0 \rangle + t(\langle 5, 5 \rangle - \langle 0, 0 \rangle) = \langle 5t, 5t \rangle, 0 \leq t \leq 1 \tag{2}$$

is a parameterization for the line segment from $(0, 0)$ to $(5, 5)$.

Solution to (2): We see that

$$\vec{r'}(t) = \langle 5, 5 \rangle \to |\vec{r'}(t)| = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}. \tag{3}$$

Solution to (3): We have

$$\int_C (x^2 + y^2) ds = \int_0^1 ((5t)^2 + (5t)^2) 5\sqrt{2} dt = 5\sqrt{2} \int_0^1 (25t^2 + 25t^2) dt \tag{4}$$

$$= 250\sqrt{2} \int_0^1 t^2 dt = \frac{250\sqrt{2}}{3} t^3 \Big|_0^1 = \boxed{\frac{250\sqrt{2}}{3}}. \tag{5}$$

Problem 2: Let $f(x,y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0,0)$ and $P(1,1)$.

(1) Let $\mathcal{C}_1$ be the segment from O to P. Find a parameterization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.

(2) Let $\mathcal{C}_2$ be the segment from P to O. Find a parameterization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.

(3) Compare the results of (1) and (2).

Solution to (1): As mentioned in the solution to problem 15.2.23, we see that $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \leq t \leq 1$ is a parameterization of the segement of the parabola $y = x^2$ from $O(0,0)$ to $P(1,1)$. We see that

$$\mathbf{r}'(t) = \langle 1, 2t\rangle \to |\mathbf{r}'(t)| = \sqrt{1^2 + 4t^2} = \sqrt{1+4t^2}, \text{ so} \tag{6}$$

$$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt - \int_0^1 f(t,t^2)\sqrt{1+4t^2}dt = \int_0^1 t\sqrt{1+4t^2}dt \tag{7}$$

$$\overset{u=1+4t^2}{=} \int_{t=0}^1 \sqrt{u}\frac{1}{8}du = \frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 = \frac{1}{12}(1+4t^2)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{5^{\frac{3}{2}}-1}{12}}. \tag{8}$$

Solution to (2): We see that if we replace t by $1-t$, then the parameterization starts at $P(1,1)$ and ends at $O(0,0)$, so $\mathbf{r}(t) = \langle 1-t, (1-t)^2\rangle$ is a parameterization of the segment of the parabola $y = x^2$ from $P(1,1)$ to $O(0,0)$. We see that

$$\mathbf{r}'(t) = \langle -1, -2(1-t)\rangle \tag{9}$$

$$\to |\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1-t))^2} = \sqrt{1+4(1-t)^2}, \text{ so} \tag{10}$$

$$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_0^1 f(1-t,(1-t)^2)\sqrt{1+4(1-t)^2}dt \tag{11}$$

$$= \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt \overset{u=1+4(1-t)^2}{=} \int_{t=0}^1 \sqrt{u}(-\frac{1}{8})du = -\frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 \tag{12}$$

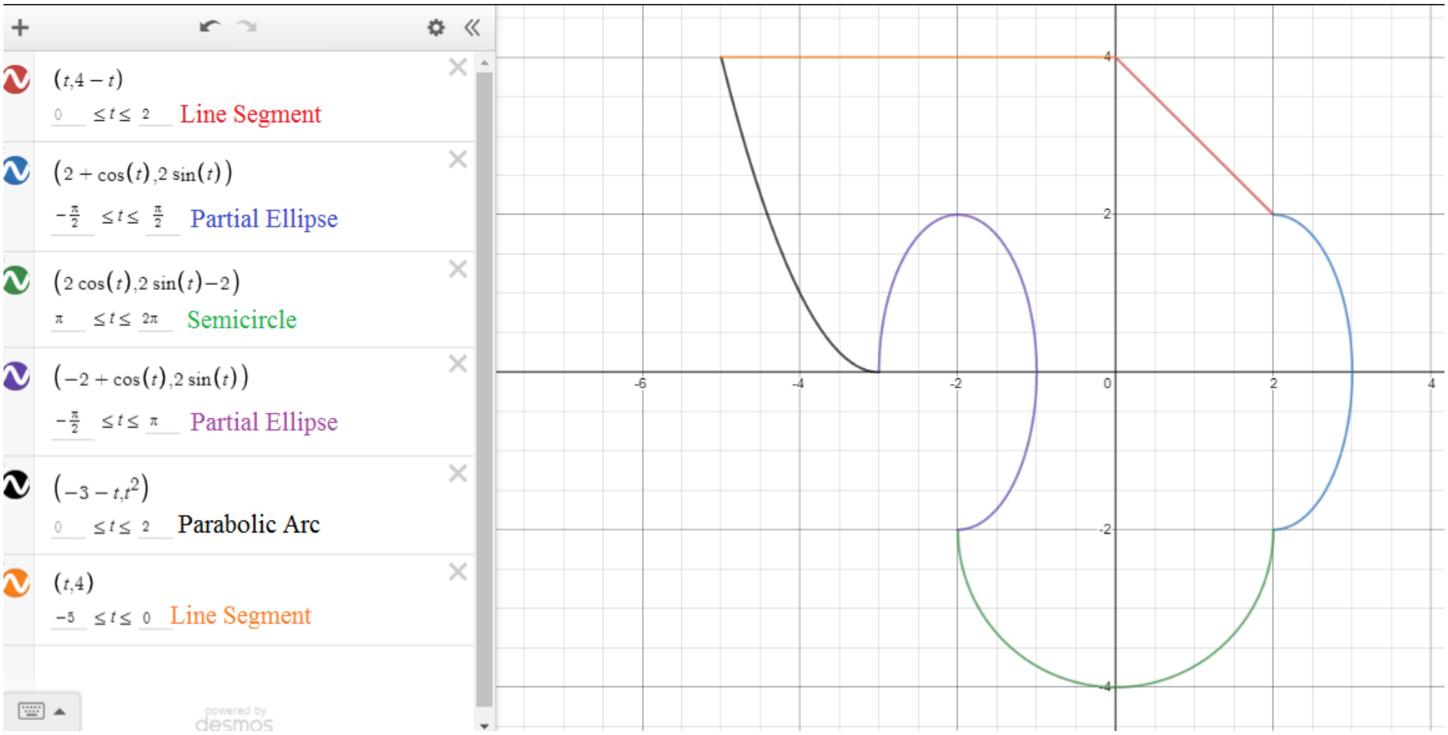
$$= -\frac{1}{12}(1+4(1-t)^2)^{\frac{3}{2}}\Big|_{t=0}^{1} = \boxed{\frac{5^{\frac{3}{2}}-1}{12}}. \tag{13}$$

Solution to (3): We see that the answers to parts (1) and (2) are the same. This makes sense because we are integrating the same function values over the same region. This should be compared to the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Note that the reason that we do not obtain a negative sign in part (2) is because $ds = |\mathbf{r}'(t)|dt$, and the absolute values absorb the negative sign. To see this fact in action back in the one dimensional case, we note that $\mathbf{r_1}(t) = \langle (b-a)t + a\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = a$ to $x = b$, and $\mathbf{r_2}(t) = \langle (a-b)t + b\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = b$ t $x = a$. We see that $\mathbf{r_1}'(t) = \langle b-a\rangle = -\langle a-b\rangle = \mathbf{r_2}'(t)$, so $ds = |\mathbf{r_1}'(t)| = |\mathbf{r_2}'(t)| = b-a$ is the same for both parameterizations.

Problem 3: Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{14}$$

where C is the curve that is shown in the picture below.



Solution: Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, \text{ and} \tag{15}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{16}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{17}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{18}$$

so $\vec{F}$ is a conservative vector field. We also see that

(19) $$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}.$$

Since $\vec{F}$ is conservative and C is a (simple piecewise smooth oriented) closed curve, we see that

(20) $$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}.$$

Challenge for the brave: Letting C once again denote the curve in figure , evaluate

(21) $$\int_C \langle y, 0 \rangle \cdot d\vec{r}.$$

Problem 4: Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

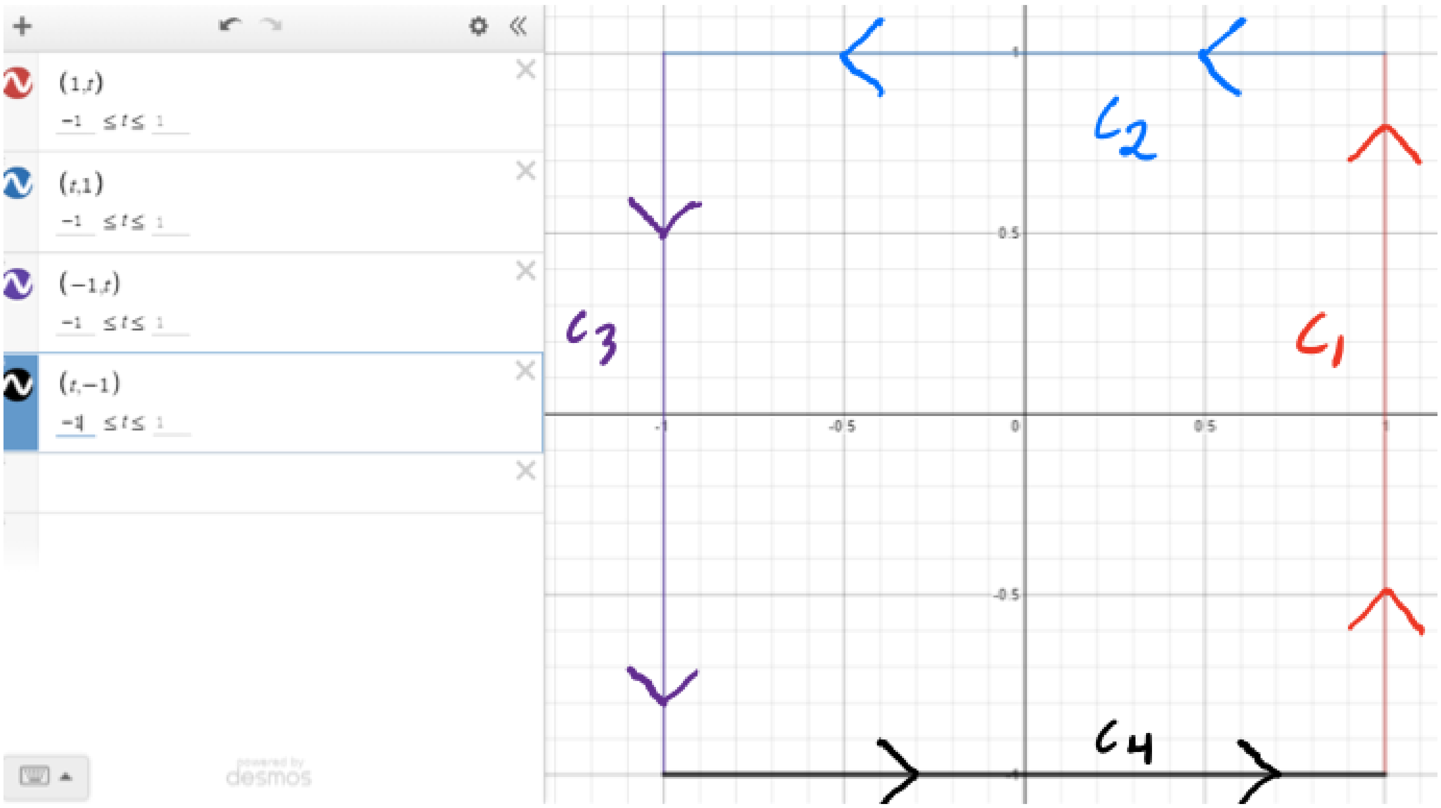


FIGURE 1. The curve C.

(a) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a parameterization $\vec{r}(t)$ for the curve C.
(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.

Solution to a: Letting C_1, C_2, C_3, and C_4 be as in Figure , we see that

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r}. \tag{22}$$

Since

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{-1}^{1}\langle 1, -t\rangle\cdot\langle 0, 1\rangle dt = \int_{-1}^{1} -t dt = -\frac{1}{2}t^2\Big|_{-1}^{1} = 0, \tag{23}$$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{1}^{-1}\langle t, -1\rangle\cdot\langle 1, 0\rangle dt = \int_{1}^{-1} t dt = \frac{1}{2}t^2\Big|_{1}^{-1} = 0, \tag{24}$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_1^{-1} \langle -1, -t \rangle \cdot \langle 0, 1 \rangle dt = \int_1^{-1} -t dt = -\frac{1}{2}t^2 \Big|_1^{-1} = 0, \tag{25}$$

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t, 1 \rangle \cdot \langle 1, 0 \rangle dt = \int_{-1}^{1} t dt = \frac{1}{2}t^2 \Big|_{-1}^{1} = 0, \tag{26}$$

we see that

$$\int_C \vec{F} \cdot d\vec{r} = 0 + 0 + 0 + 0 = \boxed{0}. \tag{27}$$

Solution to b: Since

$$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(-y), \tag{28}$$

we see that $\vec{F} = \langle x, -y \rangle$ is a conservative vector field. We now have 2 ways in which to finish the problem.

Finish 1: Since $\vec{F}$ is a conservative vector field and C is a (simple, piecewise smooth, oriented) closed curve, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{29}$$

Finish 2: We now want to find a potential function $\varphi(x, y)$ for $\vec{F}$. Since

$$\langle \varphi_x, \varphi_y \rangle = \nabla \varphi = \vec{F} = \langle x, -y \rangle, \tag{30}$$

we see that

$$\varphi_x(x, y) = x \to \varphi(x, y) = \int x dx = \frac{1}{2}x^2 + g(y) \to \tag{31}$$

$$g'(y) = \varphi_y(x, y) = -y \to g(y) = -\frac{1}{2}y^2 + C \to \varphi(x, y) = \frac{1}{2}(x^2 - y^2) + C. \tag{32}$$

Now let P be any point on the curve C. For example, we may take $P = (1,1)$. Since the curve C can be seen as starting at P and ending at P, the Fundamental Theorem for Line Integrals tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \varphi\left((1,1)\right) - \varphi\left((1,1)\right) = \boxed{0}. \tag{33}$$

Remark: We see that in Finish 2, we did not even need to determine what the function φ was in order to conclude that the final answer is 0.

Problem 5: An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R as shown in the figure below.

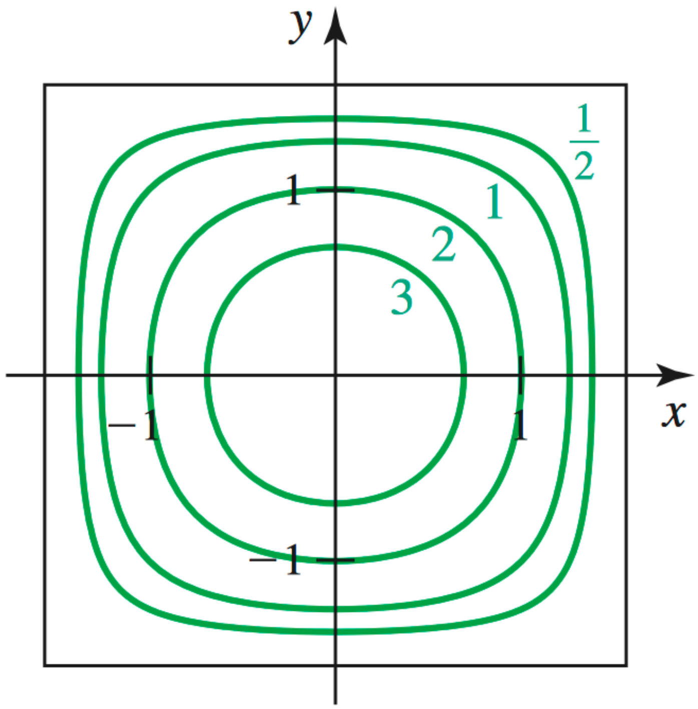


FIGURE 2. Some level curves of the stream function $\Psi(x, y)$.

(a) The horizontal (east-west) component of the velocity is $u = \Psi_y$ and the vertical (north-south) component of the velocity is $v = -\Psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.
(b) Is the velocity field source free? Explain.
(c) Is the velocity field irrotational? Explain.
(d) Find the total outward flux across $\mathcal{C}$.
(e) Find the circulation on $\mathcal{C}$ assuming counterclockwise orientation.

Solution to part (a): We see that

$$u(x, y) = \Psi_y(x, y) = -4\cos(x)\sin(y), \text{ and} \tag{34}$$

$$v(x, y) = -\Psi_x(x, y) = 4\sin(x)\cos(y), \tag{35}$$

so the velocity field $\vec{F} = \vec{F}(x, y)$ is given by

$$\vec{F}(x, y) = \langle u(x, y), v(x, y)\rangle = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y)\rangle. \tag{36}$$

Solution to part (b): We see that the divergence of $\vec{F}$ is given by

$$\text{Div}(\vec{F}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4\sin(x)\sin(y) - 4\sin(x)\sin(y) = 0, \tag{37}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is source free.}}$ In fact, we can show that any vector field $\vec{F} = \langle f, g\rangle = \langle \Psi_y, -\Psi_x\rangle$ that arises from a stream function Ψ is source free. It suffices to observe that

$$\text{Div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (\Psi_y)_x + (-\Psi_x)_y = \Psi_{yx} - \Psi_{xy} = 0. \tag{38}$$

This should be compared to the fact that any vector field $\vec{F} = \langle \varphi_x, \varphi_y\rangle$ coming from a potential function φ is conservative/irrotational.

Solution to part (c): We see that the curl of $\vec{F}$ is given by

$$\begin{aligned} \text{Curl}(F) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 4\cos(x)\cos(y) - (-4\cos(x)\cos(y)) \\ &= 8\cos(x)\cos(y) \not\equiv 0, \end{aligned} \tag{39}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is not irrotational.}}$

Solution to part (d): Using the flux form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot\hat{n}ds = \iint_R \text{Div}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0dxdy = \boxed{0}. \tag{40}$$

Solution to part (e): Using the circulation form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_R \text{Curl}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos(x)\cos(y)dxdy \tag{41}$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(y)dy\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)dx\right) = 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)dx\right)^2 \tag{42}$$

$$= 8\left(\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)^2 = 8\cdot 2^2 = \boxed{32}. \tag{43}$$

Problem 6: Consider the radial field $\vec{F}(x,y) = \frac{\langle x,y\rangle}{\sqrt{x^2+y^2}} = \frac{\vec{r}}{|\vec{r}|}$.

(a) Explain why the conditions of Green's Theorem do not apply to $\vec{F}$ on a region R containing the origin.

(b) Let R be the unit disk centered at the origin and compute

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA. \tag{44}$$

(c) Evaluate the line integral in the flux form of Green's Theorem applied to the region R and the vector field $\vec{F}$.

(d) Do the results of parts (b) and (c) agree? Explain.

Solution to part (a): We see that for

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } g(x,y) = \frac{y}{\sqrt{x^2+y^2}}, \tag{45}$$

we have $\vec{F} = \langle f, g\rangle$. One of the conditions of Green's Theorem (flux form and circulation form) is that f and g have continuous first partial derivatives in R. Since neither of f and g are continuous at $(0,0)$, their first partial derivatives don't even exist at $(0,0)$, so they are not continuous. It follows that the conditions of Green's Theorem are not satisfied if $(0,0) \in R$.

Solution to part (b): We see that

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} + x\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot 2x\right) = \frac{y^2}{\sqrt{x^2+y^2}^3}, \text{ and} \tag{46}$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} + y\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot 2y\right) = \frac{x^2}{\sqrt{x^2+y^2}^3}. \tag{47}$$

It follows that

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R \left(\frac{y^2}{\sqrt{x^2+y^2}^3} + \frac{x^2}{\sqrt{x^2+y^2}^3}\right) dA \tag{48}$$

$$= \iint_R \frac{1}{\sqrt{x^2+y^2}} dA = \int_0^{2\pi}\int_0^1 \frac{1}{r} r dr d\theta = \int_0^{2\pi}\int_0^1 dr d\theta = \boxed{2\pi}. \tag{49}$$

Solution to part (c): We recall that

$$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle, 0 \le t \le 2\pi \tag{50}$$

is the parameterization by arclength of the unit circle. In this case we may naturally identify $\vec{r}(t)$ with the radial vector $\vec{r}$, so we will do so by abuse of notation. Furthermore, recalling that $\hat{n}$ is the *outward* unit normal vector, we see that

$$\hat{n}(t) = \langle \cos(t), \sin(t)\rangle = \vec{r}(t), 0 \le t \le 2\pi. \tag{51}$$

It follows that

$$\int_{\mathcal{C}} \vec{F}\cdot\hat{n} ds = \int_{\mathcal{C}} \frac{\vec{r}}{|\vec{r}|}\cdot(\vec{r}(t)) ds = \int_0^{2\pi} \frac{|\vec{r}(t)|^2}{|\vec{r}(t)|} dt = \int_0^{2\pi} \frac{1^2}{1} dt = \boxed{2\pi}. \tag{52}$$

Solution to part (d): Even though the conditions of Green's Theorem do not apply, the answers to parts (b) and (c) are the same. This shows that the conditions of Green's Theorem are sufficient conditions but not necessary conditions to attain the result of Green's Theorem.