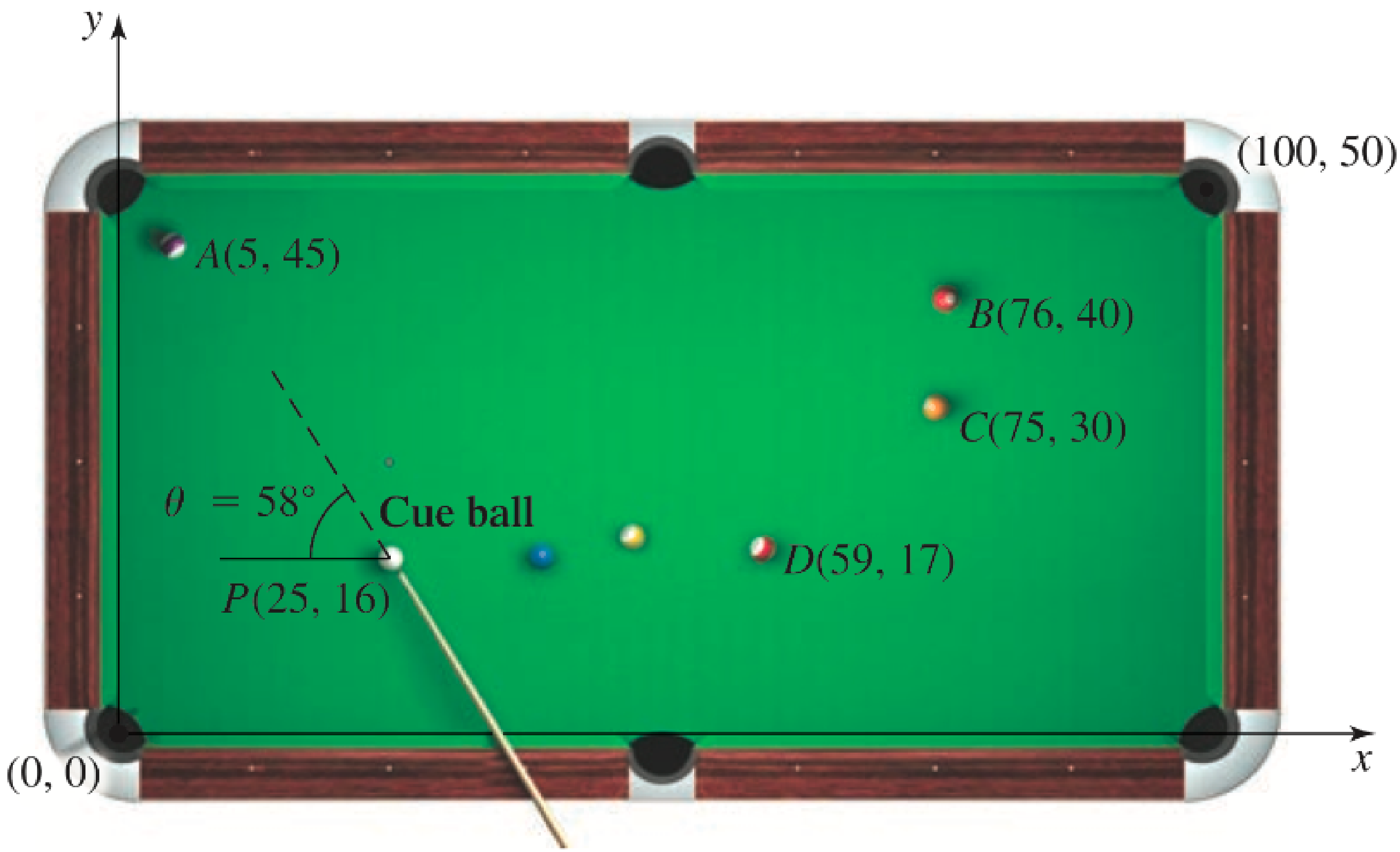
Problem 13.5.5.41: A cue ball in a billiards video game lies at $P(25, 16)$. We assume that each ball has a diameter of 2.25 screen units, and pool balls are represented by the point at their center.

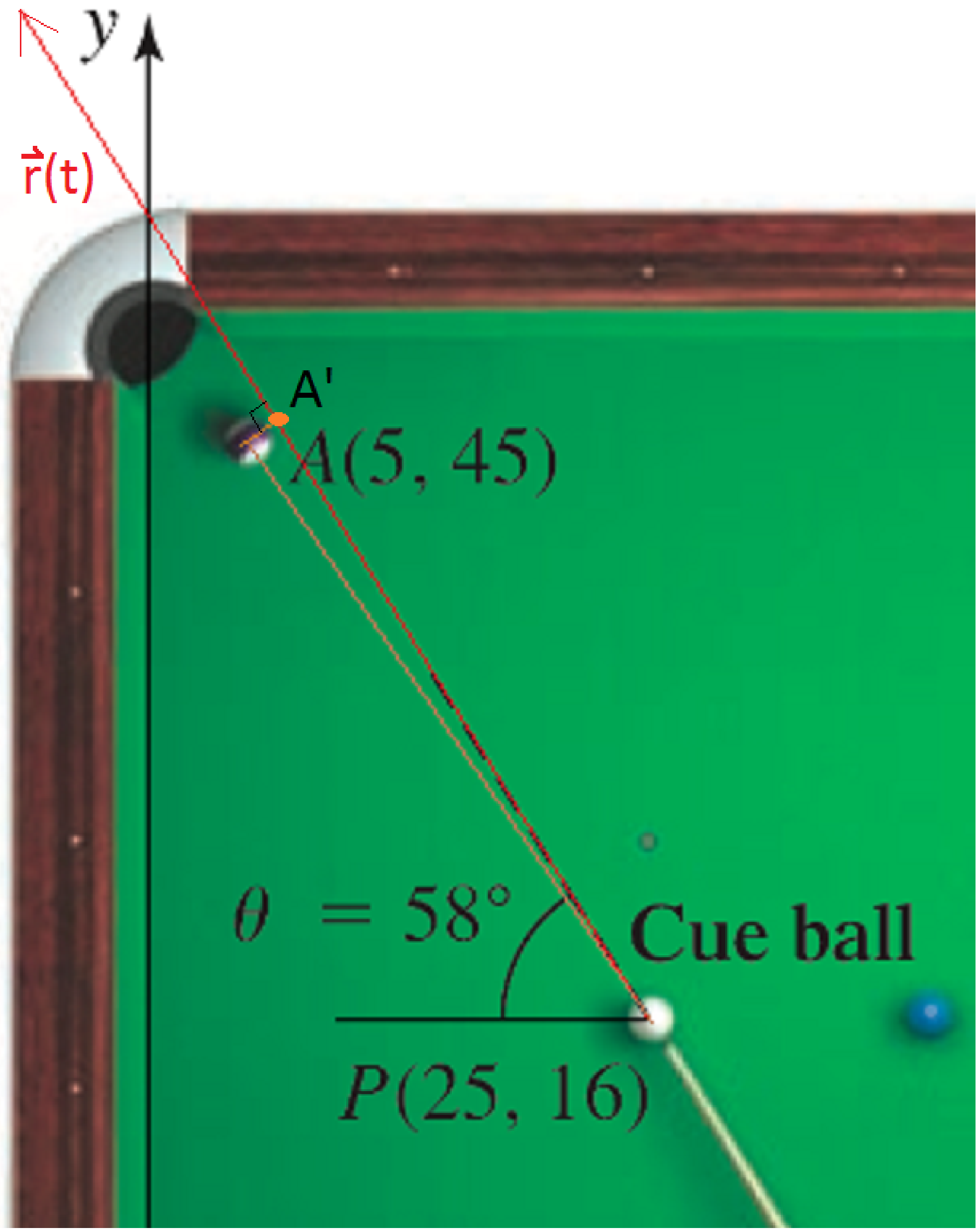
a. The cue ball is aimed at an angle of $58°$ above the negative x-axis toward a target ball at $A(5, 45)$. Do the balls collide?
b. The cue ball is aimed at the point $(50, 25)$ in an attempt to hit a target ball at $B(76, 40)$. Do the balls collide?
c. The cue ball is aimed at an angle θ above the x-axis in the general direction of a target ball at $C(75, 30)$. What range of angles (for $0 \leq \theta \leq \frac{\pi}{2}$) will result in a collision? Express your answer in degrees.



Solution to a: Since the diameter of each ball is 2.25 units, the balls collide if at some point in time their centers are at most 2.25 units away from each other. Therefore we can solve this problem by calculating the distance between the point A and the straight line that is the trajectory of the cue ball. To this end we observe that the cue ball is aimed at an angle of $180° - 58° = 122°$ above the positive x-axis, so we can paramterize the trajectory of the cue ball by

$$(1)\quad \vec{r}(t) = \langle 25, 16\rangle + t\underbrace{\langle \cos(122^\circ), \sin(122^\circ)\rangle}_{\hat{u}\text{, the direction of the trajectory}}$$
$$= \langle 25 + t\cos(122^\circ), 16 + t\sin(122^\circ)\rangle.$$

Now let A' be the point on $\vec{r}(t)$ that is closest to A, which happens to be the orthogonal projection of A onto $\vec{r}(t)$ as shown in the diagram below.



We have now reduced to problem down to whether or not $|\overrightarrow{A'A}|$ is larger than 2.25 or not. Since $\overrightarrow{A'A} = \overrightarrow{PA} - \overrightarrow{PA'}$, we first observe that

$$\overrightarrow{PA} = \underbrace{\langle 5, 45\rangle}_{\text{A}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle -20, 29\rangle, \tag{2}$$

and we will now proceed to find $\overrightarrow{PA'}$. We see that $\overrightarrow{PA'}$ is the orthogonal projection of $\overrightarrow{PA}$ onto $\vec{r}(t)$, hence

$$\overrightarrow{PA'} = (\overrightarrow{PA} \cdot \hat{u})\hat{u} \tag{3}$$

$$= \Big(\langle -20, 29\rangle \cdot \langle \cos(122^\circ), \sin(122^\circ)\rangle\Big)\langle \cos(122^\circ), \sin(122^\circ)\rangle \tag{4}$$

$$\approx \langle -18.65, 29.84\rangle. \tag{5}$$

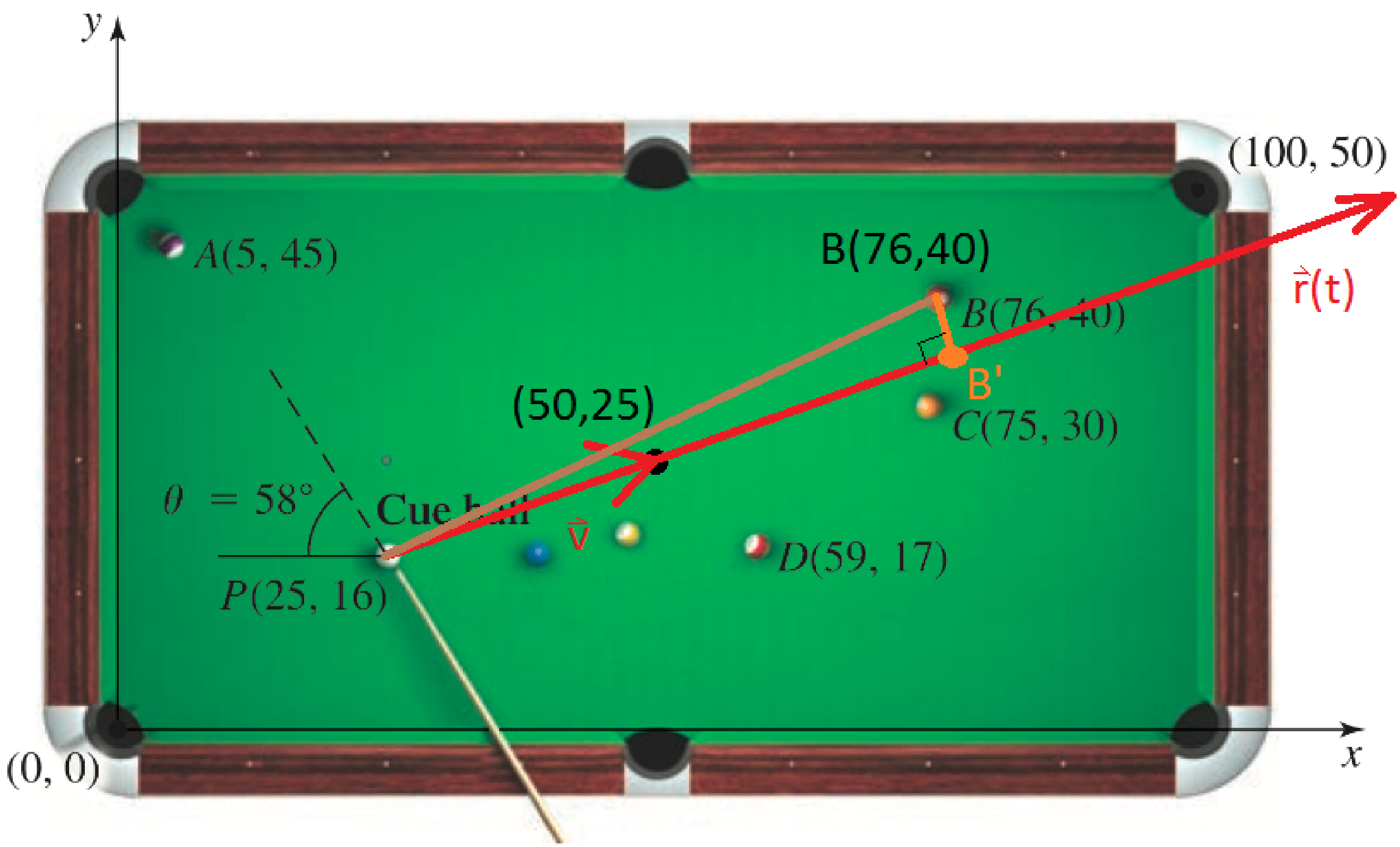
Putting everything together we see that

$$|\overrightarrow{A'A}| = |\overrightarrow{PA} - \overrightarrow{PA'}| \approx |\langle -20, 29\rangle - \langle -18.65, 29.84\rangle| \tag{6}$$

$$= |\langle -1.35, -0.84\rangle| = 1.59 < 2.25, \tag{7}$$

so the balls do collide.

Solution to b: We use the same strategy that we used in part a. The only difference is that we will use slightly different computations to obtain a parameterization of the path of the cue ball since we were given a point on its trajectory rather than the angle that the trajectory makes with the x-axis.



We see that

$$\vec{r}(t) = \langle 25, 16\rangle + t\ \underbrace{(\langle 50, 25\rangle - \langle 25, 16\rangle)}_{\text{Points in the direction of } \vec{r}(t)} \tag{8}$$

$$= \langle 25, 16\rangle + t\ \underbrace{\langle 25, 9\rangle}_{\vec{v}} = \langle 25 + 25t, 16 + 9t\rangle \tag{9}$$

Since $\overrightarrow{B'B} = \overrightarrow{PB} - \overrightarrow{PB'}$, we first observe that

$$\overrightarrow{PB} = \underbrace{\langle 76, 40\rangle}_{\text{B}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle 51, 24\rangle, \tag{10}$$

and we will now proceed to find $\overrightarrow{PB'}$. Since $\overrightarrow{PB'}$ is the orthogonal projection of $\overrightarrow{PB}$ onto $\vec{r}(t)$ we see that

$$\overrightarrow{PB'} = \frac{\overrightarrow{PB} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\langle 51, 24\rangle \cdot \langle 25, 9\rangle}{|\langle 25, 9\rangle|^2}\langle 25, 9\rangle \tag{11}$$

$$(12) \qquad = \langle \frac{37275}{706}, \frac{13419}{706} \rangle \approx \langle 52.80, 19.01 \rangle$$

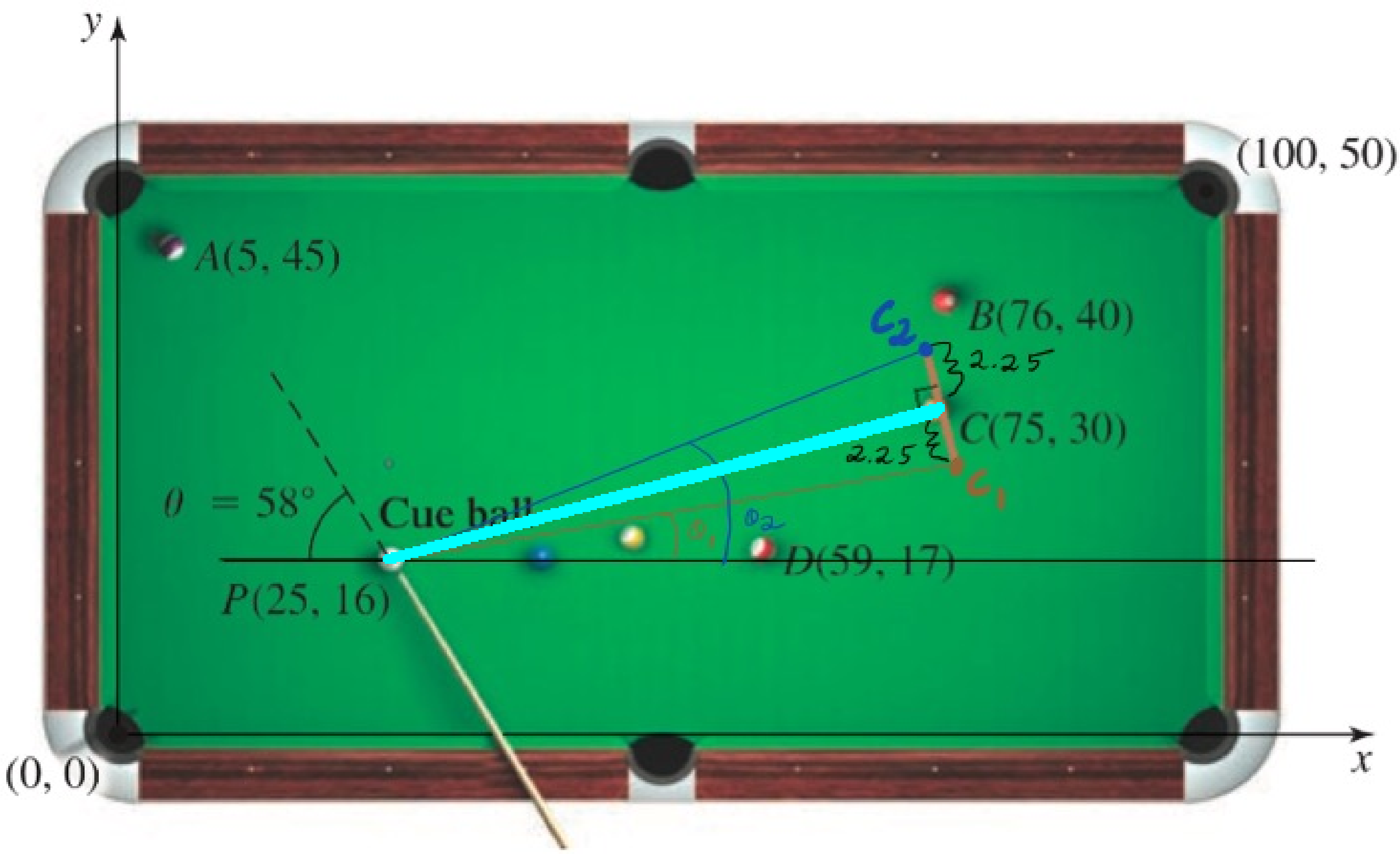
Putting everything together we see that

$$(13) \qquad |\overrightarrow{B'B}| = |\overrightarrow{PB} - \overrightarrow{PB'}| \approx |\langle 51, 24 \rangle - \langle 52.80, 19.01 \rangle|$$

$$(14) \qquad |\langle -1.80, 4.99 \rangle| \approx 5.30 > 2.25,$$

so the balls do not collide.

Solution to c: Let the points C_1 and C_2 be such that $|\overrightarrow{CC_1}| = |\overrightarrow{CC_2}| = 2.25$ and each of $\overrightarrow{CC_1}$ and $\overrightarrow{CC_2}$ are orthogonal to $\overrightarrow{PC}$. $\overrightarrow{PC_1}$ and $\overrightarrow{PC_2}$ represent the trajectories in which the cue ball just barely touches the ball at C, so we want to determine the angles θ_1 and θ_2 as shown in the diagram below.



To this end we begin by observing that

$$\overrightarrow{PC} = \underbrace{\langle 75, 30\rangle}_{C} - \underbrace{\langle 25, 16\rangle}_{P} = \langle 50, 14\rangle. \tag{15}$$

Recalling that for a given vector $\vec{w} := \langle x, y\rangle$ the vectors $\langle -y, x\rangle$ and $\langle y, -x\rangle$ are orthogonal to $\vec{w}$, we see that

$$\overrightarrow{PC_1} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle 14, -50\rangle}{|\langle 14, -50\rangle|} \approx \langle 50.61, 11.83\rangle, \text{ and} \tag{16}$$

$$\overrightarrow{PC_2} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle -14, 50\rangle}{|\langle -14, 50\rangle|} \approx \langle 49.39, 16.17\rangle. \tag{17}$$

We now see that

$$\theta_1 = \tan^{-1}\left(\frac{11.83}{50.61}\right) \approx 13.16^\circ, \text{ and} \theta_2 = \tan^{-1}\left(\frac{16.17}{49.39}\right) \approx 18.13^\circ, \tag{18}$$

so the balls will collide if $13.16^\circ \leq \theta \leq 18.13^\circ$

Modified Problem 13.5.5.31: Determine whether the lines $\vec{r}(t) = \langle 1, 3, 2\rangle + t\langle 6, -7, 1\rangle$ and $R(s) = \langle 10, 6, 14\rangle + s\langle 8, 1, 4\rangle$ are parallel or skew, and find their intersection(s) if any exist.

Solution: Let us first determine whether or not the lines paramterized by $\vec{r}(t)$ and $\vec{R}(s)$ are parallel since that requires the least computations. We see that the line paramterized by $\vec{r}(t)$ has the same direction as the vector $\langle 6, -7, 1\rangle$ and the line parameterized by $\vec{R}(s)$ has the same direction as the vector $\langle 8, 1, 4\rangle$. It is clear that there is no constant c for which

$$\langle 6, -7, 1\rangle = c\langle 8, 1, 4\rangle = \langle 8c, c, 4c\rangle \tag{19}$$

since we cannot simulataneously have $c = -7$ and $4c = 1$, so the lines in question are not parallel. Now let us search for the intersection(s) of the lines in questions while recalling that the lines will be skew if there are no intersections (since we have already shown that they are not parallel). To do this, we want to find all $t, s \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, which results in the following computations:

$$\underbrace{\langle 1, 3, 2\rangle + t\langle 6, -7, 1\rangle}_{\vec{r}(t)} = \underbrace{\langle 10, 6, 14\rangle + s\langle 8, 1, 4\rangle}_{\vec{R}(s)} \tag{20}$$

$$\Leftrightarrow \langle 1 + 6t, 3 - 7t, 2 + t\rangle = \langle 10 + 8s, 6 + s, 14 + 4s\rangle \tag{21}$$

$$\Leftrightarrow \begin{array}{rcr} 1 + 6t & = & 10 + 8s \\ 3 - 7t & = & 6 + s \\ 2 + t & = & 14 + 4s \end{array} \tag{22}$$

$$\rightarrow s = -3 - 7t \tag{23}$$

$$\rightarrow 2 + t = 14 + 4s = 14 + 4(-3 - 7t) = 2 - 28t \tag{24}$$

$$\rightarrow t = 0 \rightarrow s = -3. \tag{25}$$

However, since

$$1 + 6 \cdot 0 \neq 10 + 8 \cdot (-3), \tag{26}$$

we see that there are no $s, t \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, so the lines in question are skew.

Homemade Problem: Find an equation of the plane P through the points $\mathrm{R}(5,3,7)$, $\mathrm{S}(0,1,0)$, and $\mathrm{T}(1,2,1)$.

Solution: It will be relatively easy to find the equation of the plane P if we first find a vector $\vec{n}$ that is normal to P. To find such a $\vec{n}$ it suffices to take the cross product of any two nonparallel vectors lying in P. To this end, we see that

$$\overrightarrow{\mathrm{SR}} = \langle 5,3,7\rangle - \langle 0,1,0\rangle = \langle 5,2,7\rangle, \text{ and} \tag{27}$$

$$\overrightarrow{\mathrm{ST}} = \langle 1,2,1\rangle - \langle 0,1,0\rangle = \langle 1,1,1\rangle \tag{28}$$

are two nonparallel vectors lying in P. We now take

$$\vec{n} = \overrightarrow{\mathrm{SR}} \times \overrightarrow{\mathrm{ST}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & 7 \\ 1 & 1 & 1 \end{vmatrix} \tag{29}$$

$$\hat{i}(2\cdot 1 - 1\cdot 7) - \hat{j}(5\cdot 1 - 1\cdot 7) + \hat{k}(5\cdot 1 - 1\cdot 2) \tag{30}$$

$$= -5\hat{i} + 2\hat{j} + 3\hat{k} = \langle -5,2,3\rangle. \tag{31}$$

To derive the equation of the plane P we recall that $\vec{n}$ is perpendicular to any vector that lies in P. It follows that if (x,y,z) is an arbitrary point in P, then since $(0,1,0)$ is also a point in P the vector

$$\vec{v} := \langle x,y,z\rangle - \langle 0,1,0\rangle = \langle x,y-1,z\rangle \tag{32}$$

is a vector contained in P, so we have

$$0 = \vec{n}\cdot\vec{v} = \langle -5,2,3\rangle \cdot \langle x,y-1,z\rangle = -5x + 2y - 2 + 3z \tag{33}$$

$$\rightarrow \boxed{2 = -5x + 2y + 3z}. \tag{34}$$

Remark: We can easily check our answer by verifying that R, S, and T all satisfy equation (34). Furthermore, we see that if we replace $\vec{n}$ by $c\vec{n}$ for any nonzero constant c, then $c\vec{n}$ will still be normal to P, which will result in seemingly different equations for P such as

$$-2 = 5x - 2y - 3z \tag{35}$$

when $c = -1$. In particular the order of the cross product in equation (29) and the order of the subtraction in equations (27) and (28) don't matter since the end result will at worst alter $\vec{n}$ by a negative sign.