Problem 1: Consider
$$\int_{\mathcal{C}} (x^2 + y^2)\,ds,$$
where $\mathcal{C}$ is the line segment from $(0,0)$ to $(5,5)$.

(1) Find a parametric description for $\mathcal{C}$ in the form $\vec{r}(t) = \langle x(t), y(t) \rangle$. (*Remember to state the domain of the parameter.*)

(2) Evaluate $|\vec{r}'(t)|$. $ds = |\vec{r}'(t)|\,dt$

(3) Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.

$\vec{P} = \langle 0,0 \rangle$, $\vec{Q} = \langle 5,5 \rangle$

$$\vec{r}(t) = \vec{P} + (\underbrace{\vec{Q}-\vec{P}}_{\text{"direction"}})t, \quad 0 \le t \le 1$$

$$= \langle 0,0 \rangle + (\langle 5,5 \rangle - \langle 0,0 \rangle)t$$

$$\boxed{= \langle \underbrace{5t}_{x(t)}, \underbrace{5t}_{y(t)} \rangle, \quad 0 \le t \le 1.}$$

2) $\vec{r}'(t) = \langle 5,5 \rangle$, $0 \le t \le 1$.

$$|\vec{r}'(t)| = \sqrt{5^2+5^2} = 5\sqrt{2}$$

$$\int_C (x^2+y^2)\,ds = \int_0^1 (x(t)^2 + y(t)^2)\,|\vec{r}'(t)|\,dt$$

$$= \int_0^1 ((5t)^2 + (5t)^2)\,5\sqrt{2}\,dt = 125\sqrt{2}\int_0^1 2t^2\,dt$$

$$= \frac{250\sqrt{2}}{3} t^3 \Big|_0^1 = \boxed{\frac{250\sqrt{2}}{3}}$$

Problem 2: Let $f(x,y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0,0)$ and $P(1,1)$.

(1) Let $\mathcal{C}_1$ be the segment from O to P. Find a parameterization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.

(2) Let $\mathcal{C}_2$ be the segment from P to O. Find a parameterization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.

(3) Compare the results of (1) and (2).

Problem 3: Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{1}$$

where C is the curve that is shown in the picture below.

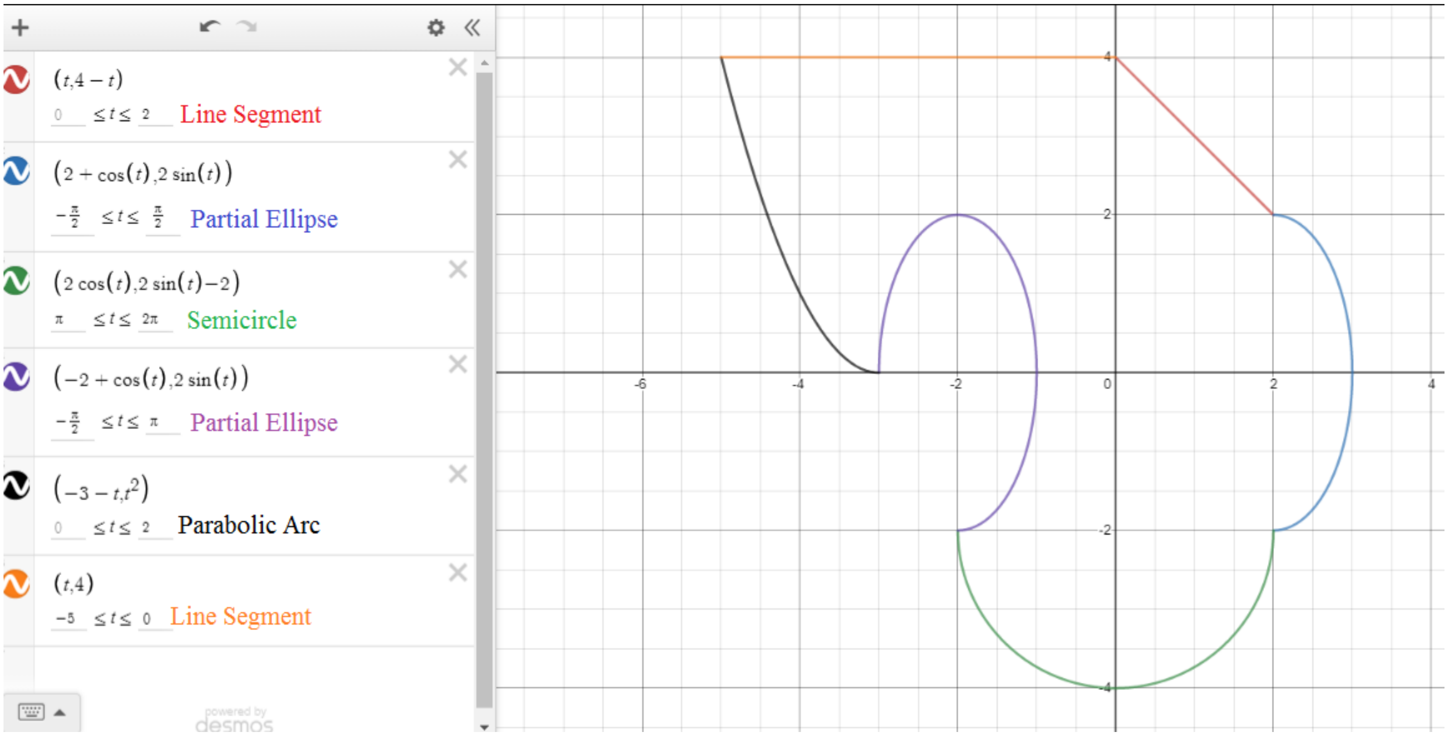


FIGURE 1

Problem 4: Consider the vector field $\vec{F} = \langle x, -y \rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

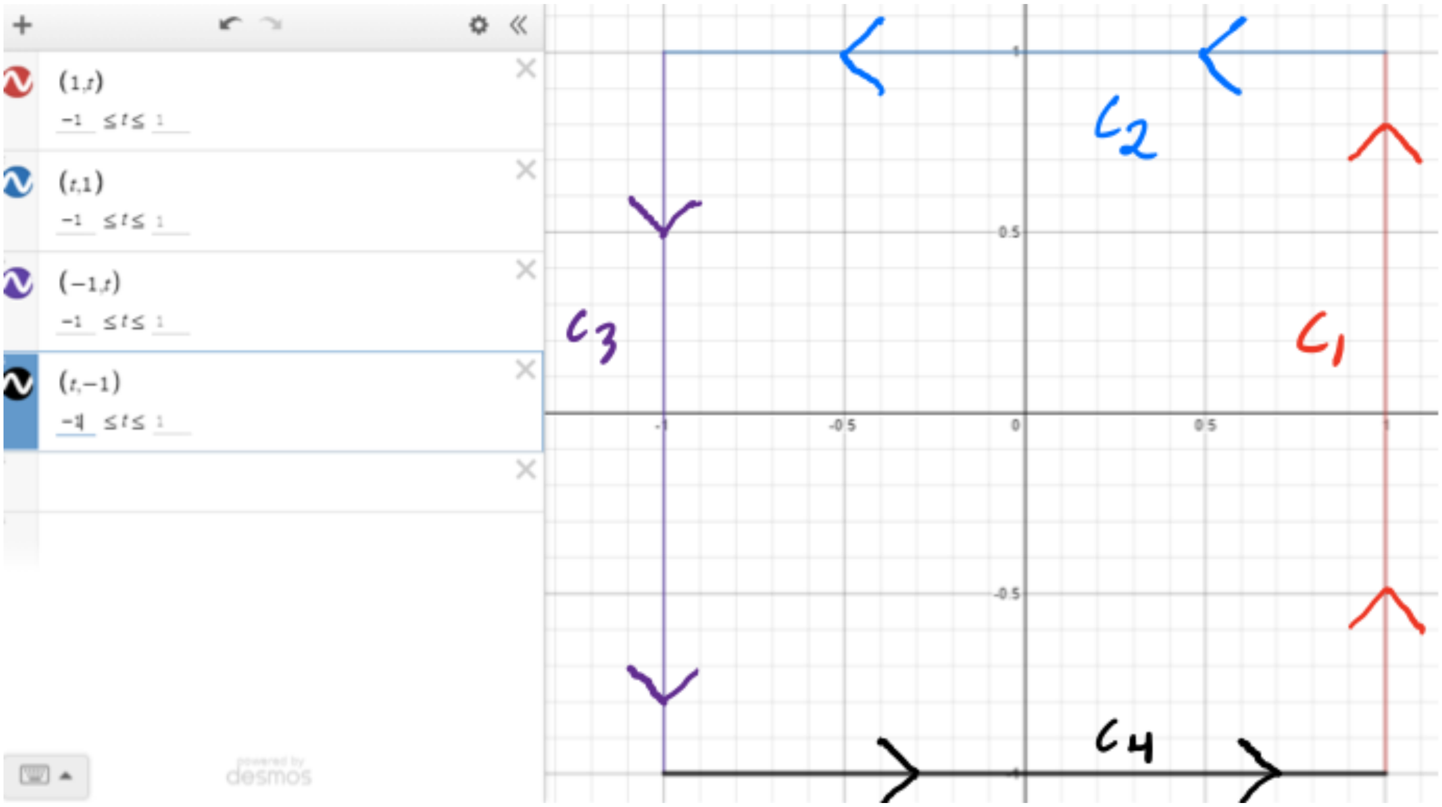


FIGURE 2. The curve C.

(a) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by finding a parameterization $\vec{r}(t)$ for the curve C.
(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.

Problem 5: An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R as shown in the figure below.

$(x,y) = (\frac{\pi}{3}, \frac{\pi}{3})$

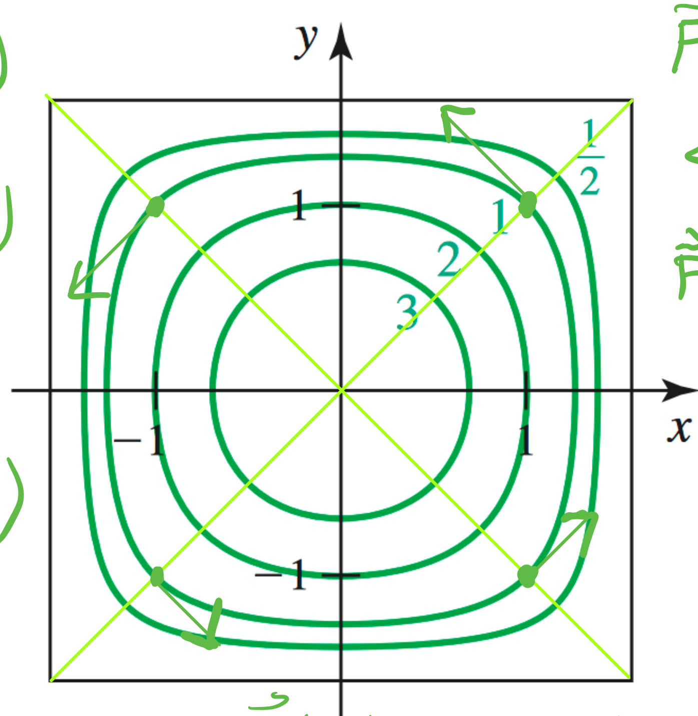
$= (-\frac{\pi}{3}, -\frac{\pi}{3})$

$= (-\frac{\pi}{3}, \frac{\pi}{3})$

$= (\frac{\pi}{3}, -\frac{\pi}{3})$

$\vec{F}(\frac{\pi}{3}, \frac{\pi}{3}) = \langle -\sqrt{3}, \sqrt{3} \rangle$

$\vec{F}(-\frac{\pi}{3}, -\frac{\pi}{3}) = \langle \sqrt{3}, -\sqrt{3} \rangle$



$\vec{F}(x,y) = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y) \rangle.$

FIGURE 3. Some level curves of the stream function $\Psi(x, y)$.

(a) The horizontal (east-west) component of the velocity is $u = \Psi_y$ and the vertical (north-south) component of the velocity is $v = -\Psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.

(b) Is the velocity field source free? Explain.

(c) Is the velocity field irrotational? Explain.

(d) Find the total outward flux across $\mathcal{C}$.

(e) Find the circulation on $\mathcal{C}$ assuming counterclockwise orientation.

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a) $\vec{F}(x,y) = \langle u(x,y), v(x,y) \rangle$

$= \langle \Psi_y, -\Psi_x \rangle$

$= \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y) \rangle.$

b) $\vec{F} = \langle \underbrace{\psi_y}_{f}, \underbrace{-\psi_x}_{g} \rangle \rightarrow$

$$\mathrm{Div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (\psi_y)_x + (-\psi_x)_y$$

$$= \psi_{yx} - \psi_{xy} = 0 \rightarrow$$

If $\vec{F}$ has a stream function, then $\vec{F}$ is source free.

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c) No. The only $\vec{F}$ that is source free <u>AND</u> irrotational is $\vec{F}(x,y) = \langle 0,0 \rangle$.

$$\vec{F}(x,y) = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y) \rangle$$

$$\mathrm{Curl}(\vec{F}) = \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

$$= 4\cos(x)\cos(y) - (-4\cos(x)\cos(y))$$

$$= 8\cos(x)\cos(y) \neq 0$$

$\rightarrow \vec{F}$ is <u>not</u> irrotational

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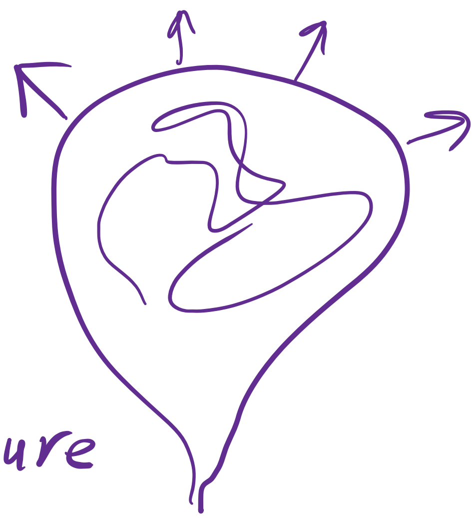
d)

$$\text{Flux}_C(\vec{F}) = \int_C \vec{F}\cdot\hat{n}\,ds \quad (\text{Flux Form of Green's Theorem})$$

$\boxed{\vec{F} = \langle f, g \rangle}$

$$= \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}_{\text{Div}(\vec{F})} dA$$

$$= \iint_R 0\,dA = \boxed{0}$$



The air in a balloon puts outward pressure on the surface of the balloon.

A Flux integral tells you the exact amount of pressure on the surface of the balloon.

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e)

$$\text{Circulation}_C(\vec{F}) = \int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

(by the circulation form of Green's Theorem)

$$= \iint_R \text{Curl}(\vec{F})\,dA$$

$$= \iint_R 8\cos(x)\cos(y)\,dA$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos(x)\cos(y)\,dx\,dy$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y)\,dy\right)$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\,dx\right)^2 = 8\left(\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)^2$$

$$= 8(1-(-1))^2 = 8\cdot 2^2 = \boxed{32}$$

Problem 6: Consider the radial field $\vec{F}(x,y) = \frac{\langle x,y\rangle}{\sqrt{x^2+y^2}} = \frac{\vec{r}}{|\vec{r}|}$.

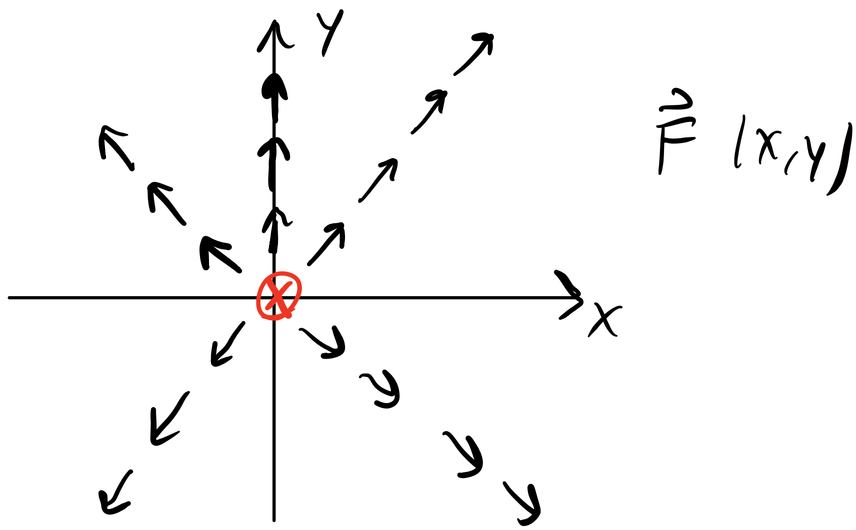
(a) Explain why the conditions of Green's Theorem do not apply to $\vec{F}$ on a region R containing the origin.

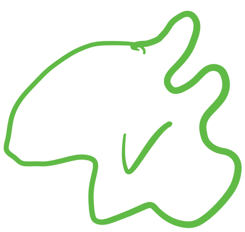
(b) Let R be the unit disk centered at the origin and compute

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA. \tag{2}$$

(c) Evaluate the line integral in the flux form of Green's Theorem applied to the region R and the vector field $\vec{F}$.

(d) Do the results of parts (b) and (c) agree? Explain.





Flux Form of Green's Theorem: If C is a closed piecewise-smooth curve with connected and simply connected interior R, and $\vec{F} = \langle f, g\rangle$ has f, g with continuous partial derivatives on R, then

$$\mathrm{Div}_C(\vec{F}) = \int_C \vec{F}\cdot\hat{n}\,ds = \iint_R \overbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}^{\mathrm{Div}(\vec{F})} dA.$$

f and g are not even defined at (0,0), so their partial derivatives are not continuous on all of R if $(0,0) \in R$.

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b) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$

$$\rightarrow \frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} + x\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot 2x\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{\sqrt{x^2+y^2}^3}$$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2}^3} - \frac{x^2}{\sqrt{x^2+y^2}^3} = \frac{y^2}{\sqrt{x^2+y^2}^3}.$$

$$g(x,y) = \frac{y}{\sqrt{x^2+y^2}} \rightarrow \frac{\partial g}{\partial y} = \frac{x^2}{\sqrt{x^2+y^2}^3}$$

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R \left(\frac{y^2}{\sqrt{x^2+y^2}^3} + \frac{x^2}{\sqrt{x^2+y^2}^3}\right) dA$$

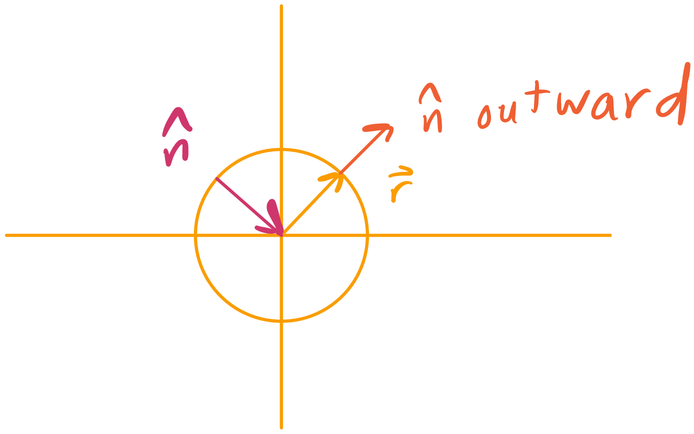
$$= \iint_R \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}}\, dA = \iint_R \frac{1}{(x^2+y^2)^{\frac{1}{2}}}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{r}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 1\,dr\,d\theta = \boxed{2\pi}$$

c) $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \ 0 \le t \le 2\pi.$

$\hat{n}(t) = \langle \cos(t), \sin(t) \rangle = \vec{r}(t).$

outward unit normal vector.



$$\text{Flux}_C(\vec{F}) = \int_C \vec{F} \cdot \hat{n}\, ds$$

$$= \int_0^{2\pi} \frac{\vec{r}(t)}{|\vec{r}(t)|} \cdot \vec{r}(t) \cdot dt$$

$$= \int_0^{2\pi} \frac{|\vec{r}(t)|^2}{|\vec{r}(t)|}\, dt$$

$$= \int_0^{2\pi} \frac{1^2}{1}\, dt = \boxed{2\pi}$$

d) The answers to parts b and c are the same, so the conditions of Green's Theorem are not always necessary even though they are sufficient.