Problem 1: Let R be the region in the xy-plane that is bounded by the spiral $r = \theta$ for $0 \le \theta \le \pi$ and the x-axis. Find the volume of the 3-dimensional solid S that lies above the region R and underneath the surface $z = x^2 + y^2$.

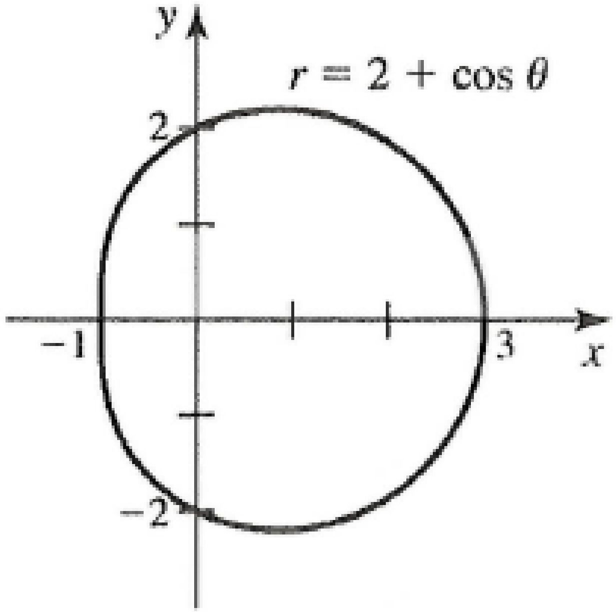
Solution: Below is a picture of the region R, which is the base of our solid S.

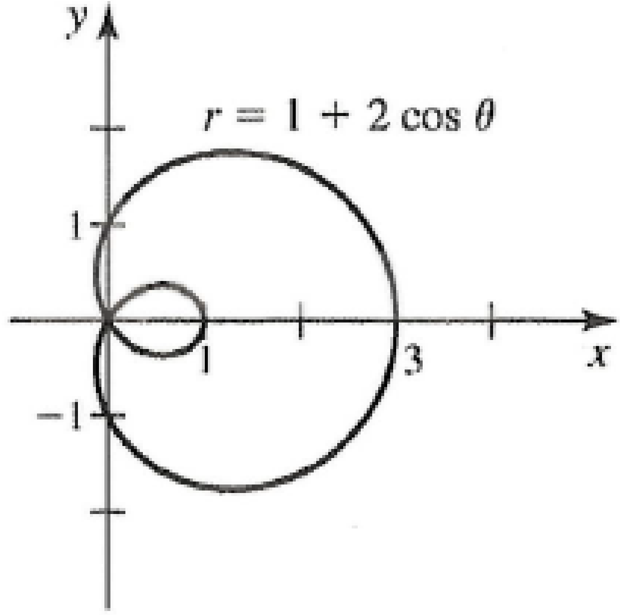
(1) $$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R \underbrace{(x^2+y^2) - 0}_{r^2} \underbrace{dA}_{r dr d\theta}$$

(2) $$= \int_0^{\pi} \int_0^{\theta} r^3 dr d\theta = \int_0^{\pi} \frac{1}{4} r^4 \Big|_{r=0}^{\theta} d\theta = \int_0^{\pi} \frac{1}{4}\theta^4 d\theta$$

(3) $$= \frac{1}{20}\theta^5 \Big|_0^{\pi} = \boxed{\frac{\pi^5}{20}}.$$

Problem 2: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.





(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.
(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.
(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Solution to (a): Letting R denote the interior of the limaçon $r = 2+\cos(\theta)$, we see that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} rdrd\theta \tag{4}$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi} \frac{1}{2}(2+\cos(\theta))^2 d\theta \tag{5}$$

$$= \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{4}\cos(2\theta)+\frac{1}{4})d\theta \tag{6}$$

$$(\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}. \tag{7}$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{8}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos(\theta))^2 d\theta \tag{9}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + 2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + \cos(2\theta) + 1)d\theta \tag{10}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}. \tag{11}$$

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{12}$$

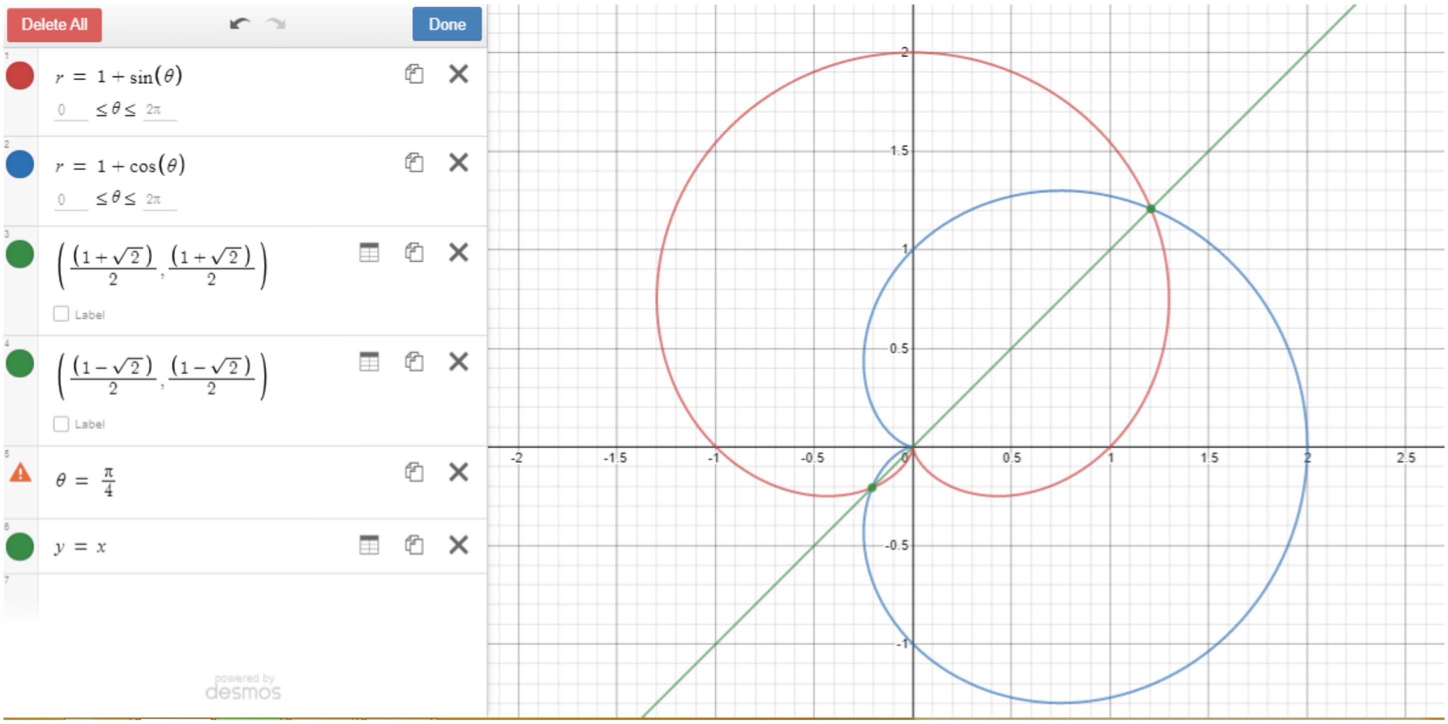
$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{13}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{14}$$

Problem 3: Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1+\cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.

Solution: We begin by drawing a picture of the region R.



We see that the 2 cardiods intersect when $\sin(\theta) = \cos(\theta)$, which occurs when $\theta = \frac{\pi}{4}, -\frac{3\pi}{4}$. We now see that when $-\frac{3\pi}{4} \le \theta \le \frac{\pi}{4}$ we have $1 + \sin(\theta) \le 1 + \cos(\theta)$ and when $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$ we have $1 + \cos(\theta) \le 1 + \sin(\theta)$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_0^{1+\sin(\theta)} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos(\theta)} rdrd\theta \tag{15}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\sin(\theta)} d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\cos(\theta)} d\theta \tag{16}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \sin^2(\theta))d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \cos^2(\theta))d\theta \tag{17}$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \frac{1-\cos(2\theta)}{2})d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2})d\theta \tag{18}$$

$$= \left(\frac{3}{4}\theta - \cos(\theta) + \frac{-\sin(2\theta)}{4}\Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\right) + \left(\frac{3}{4}\theta + \sin(\theta) + \frac{\sin(2\theta)}{4}\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right) \tag{19}$$

$$= \boxed{\frac{3\pi}{2} - 2\sqrt{2}}. \tag{20}$$

Problem 4: Write an iterated integral for $\iiint_D f(x,y,z)dV$, where D is a sphere of radius 9 centered at $(0,0,1)$. Use the order $dV = dzdydx$.

Hint: Start by finding the equation of the the surface of the sphere of radius 9 *centered at* $(0,0,1)$.

Solution: We recall that the equation of the sphere of radius R centered at (a,b,c) is given by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2, \tag{21}$$

so the equation of the sphere of radius 9 centered at $(0,0,1)$ is given by

$$x^2+y^2+(z-1)^2=81. \tag{22}$$

Since we are considering

$$\iiint_D f(x,y,z)dV=\int_?^?\int_?^?\int_?^? f(x,y,z)dzdydx, \tag{23}$$

we begin by observing that the smallest possible value of x in our region D is -9, and the largest possible value of x in our region D is 9, so $-9\le x\le 9$. We then observe that for each $x\in[-9,9]$, we have

$$y^2+(z-1)^2=81-x^2, \tag{24}$$

so the smallest possible value of y in our region D (corresponding to our chosen value of x) is $-\sqrt{81-x^2}$ and the largest possible value of y in our region D (corresponding to our chosen value of x) is $\sqrt{81-x^2}$, so $-\sqrt{81-x^2}\le y\le\sqrt{81-x^2}$. Lastly, we observe that for each $x\in[-9,9]$ and each $y\in[-\sqrt{81-x^2},\sqrt{81-x^2}]$, we have

$$(z-1)^2=81-x^2-y^2\to z=1\pm\sqrt{81-x^2-y^2}, \tag{25}$$

so the smallest possible value of z in our region D (corresponding to our chosen values of x and y) is $1-\sqrt{81-x^2-y^2}$ and the largest possible value of y in our region D (corresponding to our chosen values of x and y) is $1+\sqrt{81-x^2-y^2}$, so $1-\sqrt{81-x^2-y^2} \le z \le 1+\sqrt{81-x^2-y^2}$. It follows that we can describe our region D as

$$D = \Big\{(x,y,z) : -9 \le x \le 9, -\sqrt{81-x^2} \le y \le \sqrt{81-x^2}, 1-\sqrt{81-x^2-y^2} \le z \le 1+\sqrt{81-x^2-y^2}\Big\}, \text{ so} \tag{26}$$

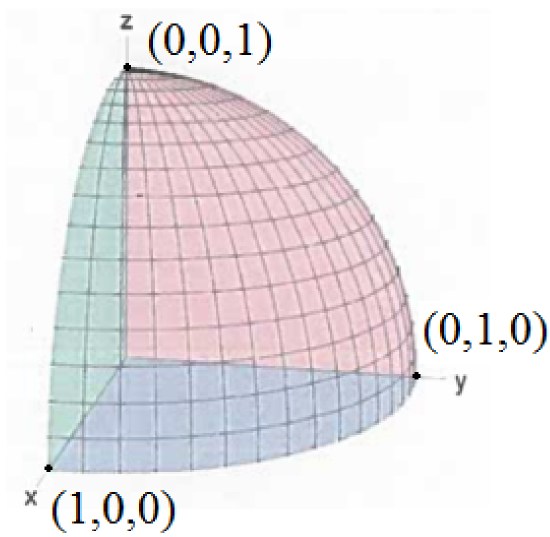
$$\iiint_D f(x,y,z)dV = \boxed{\int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{1-\sqrt{81-x^2-y^2}}^{1+\sqrt{81-x^2-y^2}} f(x,y,z)dzdydx}. \tag{27}$$

Problem 5: Sketch by hand or graph with a computer program the region of integration for the integral

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x,y,z)dxdydz. \tag{28}$$

Note: You may also describe the region of integration in writing instead. If you choose to do this, please write complete sentences and provide a thorough description.

Solution: Since $x^2 + y^2 + z^2 = 1$ is the equation of the unit sphere (centered at (0,0,0)), we may repeat the steps of the previous problem to see that the region of integration is related to the unit sphere. The key difference here is that the smallest possible values of x, y, and z are always 0, so our region of integration ends up being the portion of the unit sphere within the first octant.



Problem 6: Evaluate

$$\int_1^{\ln(8)}\int_1^{\sqrt{z}}\int_{\ln(y)}^{\ln(2y)} e^{x+y^2-z}dxdydz. \tag{29}$$

Solution: We see that

$$\int_1^{\ln(8)}\int_1^{\sqrt{z}}\int_{\ln(y)}^{\ln(2y)} e^{x+y^2-z}dxdydz = \int_1^{\ln(8)}\int_1^{\sqrt{z}} e^{x+y^2-z}\Big|_{x=\ln(y)}^{\ln(2y)}dydz \tag{30}$$

$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}}(e^{\ln(2y)+y^2-z} - e^{\ln(y)+y^2-z})dydz \tag{31}$$

$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}}(2ye^{y^2-z} - ye^{y^2-z})dydz \tag{32}$$

$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}} ye^{y^2-z}dydz \overset{u=y^2}{=} \int_1^{\ln(8)}\frac{1}{2}e^{y^2-z}\Big|_{y=1}^{\sqrt{z}} \tag{33}$$

$$\frac{1}{2}\int_1^{\ln(8)}(e^0 - e^{1-z})dz = \frac{1}{2}(z + e^{1-z}\Big|_1^{\ln(8)}) \tag{34}$$

$$= \frac{1}{2}(\ln(8) + e^{1-\ln(8)} - (e^{1-1}+1)) \tag{35}$$

$$= \frac{1}{2}(\ln(8)+e^1\cdot e^{-\ln(8)}-e^0-1) = \frac{1}{2}(\ln(8)+\frac{e}{e^{\ln(8)}}-2) = \boxed{\frac{1}{2}\ln(8)+\frac{e}{16}-1}. \tag{36}$$