Problem 1: Suppose that the second partial derivative of f are continuous on $R = \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}$. Show that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = f(a,b) - f(a,0) - f(0,b) + f(0,0). \tag{1}$$

Hint: Think about the fundamental theorem of calculus.

Solution: We see that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = \int_0^b \int_0^a \frac{\partial^2 f}{\partial x \partial y}(x,y)dxdy = \int_0^b \frac{\partial f}{\partial y}(x,y)\Big|_{x=0}^{a} dy \tag{2}$$

$$= \int_0^b \left(\frac{\partial f}{\partial y}(a,y) - \frac{\partial f}{\partial y}(0,y) \right) dy = (f(a,y) - f(0,y))\Big|_0^b. \tag{3}$$

$$= f(a,b) - f(a,0) - f(0,b) + f(0,0). \tag{4}$$

Alternatively, since the second partial derivatives of f are continuous on R, we can use Clairaut's Theorem to perform the calculations in the following fashion.

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = \int_0^a \int_0^b \frac{\partial^2 f}{\partial y \partial x}(x,y)dydx = \int_0^a \frac{\partial f}{\partial x}(x,y)\Big|_{y=0}^{b} dx \tag{5}$$

$$= \int_0^a \left(\frac{\partial f}{\partial x}(x,b) - \frac{\partial f}{\partial x}(x,0) \right) dx = (f(x,b) - f(x,0))\Big|_0^a. \tag{6}$$

$$= f(a,b) - f(a,0) - f(0,b) + f(0,0). \tag{7}$$

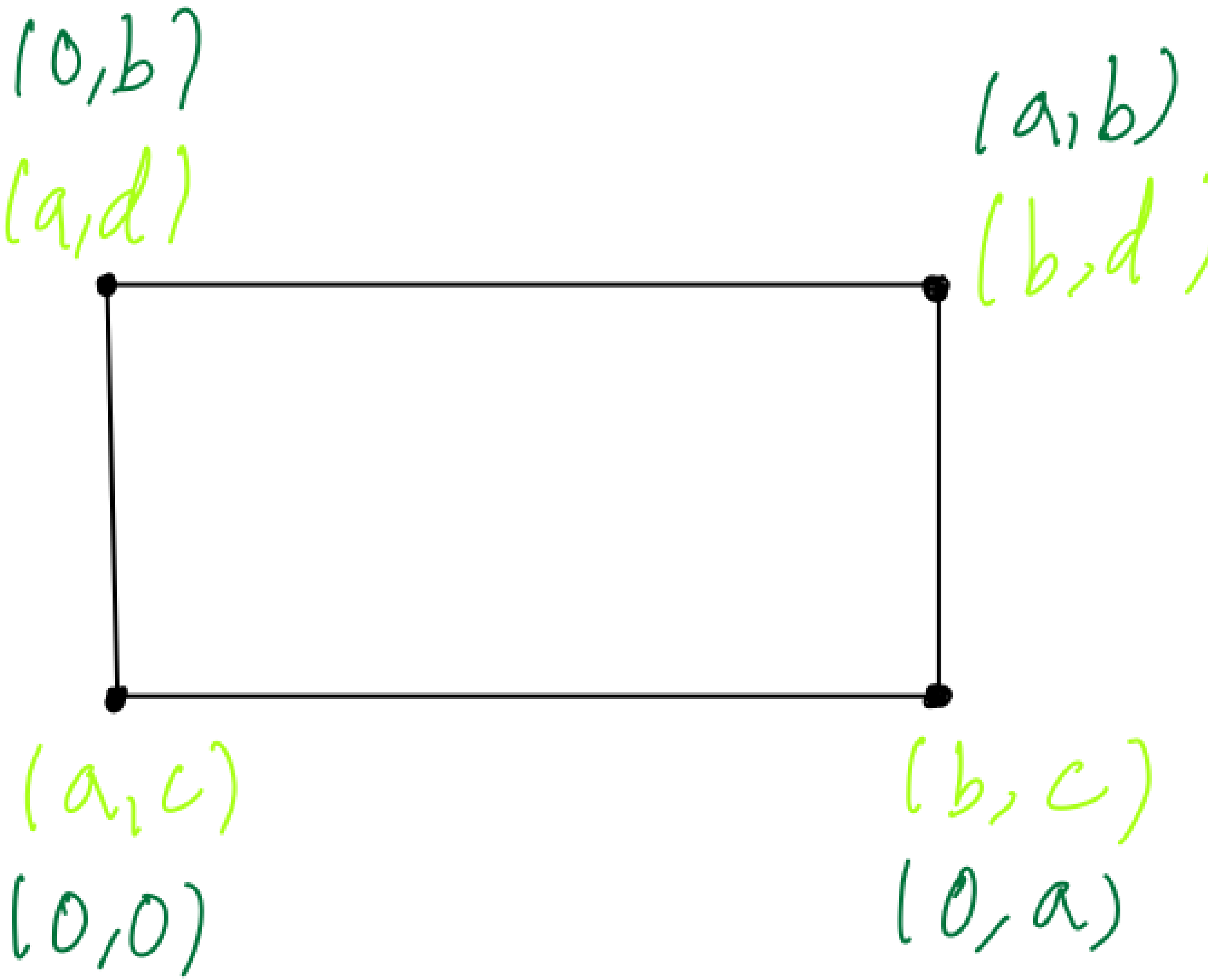
Remark: A similar method can show that if $R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x,y)dA = f(b,d) - f(a,d) - f(b,c) + f(a,c). \tag{8}$$

The Fundamental Theorem of Calculus told us that

$$\int_a^b \frac{df}{dx}(x)dx = f(b) - f(a). \tag{9}$$

Comparing equations (9) and (8), we see that instead taking the difference at the 2 endpoints of a line segment, we are adding 2 opposite corners of the rectangular region R ($f(b,d)$ and $f(a,c)$, or $f(a,b)$ and $f(0,0)$ from the original problem) and subtracting from that the sum of the other 2 opposite corners ($f(a,d)$ and $f(b,c)$, or $f(a,0)$ and $f(0,b)$ from the original problem).



Problem 2: Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

a. Evaluate $\iint_R \cos(x\sqrt{y})dA$.

b. Evaluate $\iint_R x^3 y \cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

Solution to a: Noting that $\int \cos(cx)dx$ is easily computable, but $\int \cos(c\sqrt{y})dy$ is not easily computable, we decide to use the order of integration given by $dA = dxdy$. It follows that

(10) $$\iint_R \cos(x\sqrt{y})dA = \int_0^1 \int_0^1 \cos(x\sqrt{y})dxdy$$

(11) $$\overset{u=x\sqrt{y}}{=} \int_0^1 \int_0^1 \frac{\cos(x\sqrt{y})}{\sqrt{y}}\sqrt{y}dxdy \overset{u=x\sqrt{y}}{=} \int_0^1 \int_{x=0}^1 \frac{\cos(u)}{\sqrt{y}}dudy$$

(12) $$= \int_0^1 \left(\frac{\sin(u)}{\sqrt{y}}\Big|_{x=0}^1\right) dy = \int_0^1 \left(\frac{\sin(x\sqrt{y})}{\sqrt{y}}\Big|_{x=0}^1\right) dy = \int_0^1 \frac{\sin(\sqrt{y})}{\sqrt{y}}dy$$

(13) $$\overset{u=\sqrt{y}}{=} \int_0^1 2\sin(\sqrt{y})\frac{dy}{2\sqrt{y}} \overset{u=\sqrt{y}}{=} \int_{y=0}^1 2\sin(u)du = -2\cos(u)\Big|_{y=0}^1$$

(14) $$= -2\cos(\sqrt{y})\Big|_{y=0}^1 = \boxed{2 - 2\cos(1)}.$$

Solution to b: Noting that $\int c_1x^3\cos(c_2x^2)dx$ is not easily computable, but $\int c_1y\cos(c_2y^2)dy$ is easily computable, we decide to use the order of integration given by $dA = dydx$. It follows that

(15) $$\iint_R x^3y\cos(x^2y^2)dA = \int_0^1 \int_0^1 x^3y\cos(x^2y^2)dydx$$

$$\overset{u=y^2}{=} \int_0^1 \int_0^1 \frac{x^3}{2}\cos(x^2y^2)2ydydx \overset{u=y^2}{=} \int_0^1 \int_{y=0}^1 \frac{x^3}{2}\cos(x^2u)dudx \tag{16}$$

$$\overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(x^2u)x^2dudx \overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(v)dvdx \tag{17}$$

$$= \int_0^1 \left(\frac{x}{2}\sin(v)\Big|_{y=0}^1\right)dx = \int_0^1 \left(\frac{x}{2}\sin(x^2u)\Big|_{y=0}^1\right)dx \tag{18}$$

$$= \int_0^1 \left(\frac{x}{2}\sin(x^2y^2)\Big|_{y=0}^1\right)dx = \int_0^1 \frac{x}{2}\sin(x^2)dx \overset{u=x^2}{=} \int_0^1 \frac{1}{4}\sin(x^2)2xdx \tag{19}$$

$$\overset{u=x^2}{=} \int_{x=0}^1 \frac{1}{4}\sin(u)du = -\frac{1}{4}\cos(u)\Big|_{x=0}^1 \tag{20}$$

$$= -\frac{1}{4}\cos(x^2)\Big|_{x=0}^1 = \boxed{\frac{1}{4} - \frac{1}{4}\cos(1)}. \tag{21}$$

Problem 3: Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let F be an antiderivative of f satisfying $F(0) = 0$, and let G be an antiderivative of F. Show that if f and F are integrable, and $r, s \ge 1$ are real numbers, then

$$\iint_R x^{2r-1}y^{s-1}f(x^ry^s)dA = \frac{G(1)-G(0)}{rs}. \tag{22}$$

Hint: Pick a convenient order of integration, then apply u-substitition. It also helps if you do part b of Problem 2 before doing this problem.

Solution: We note that part b of Problem 2 was a special instance of this problem in which $r = s = 2$ and $f(t) = \cos(t)$. Therefore we will proceed in a similar fashion, but we will slightly simplify our solution by merging the first 2 u-substitutions that were performed in the solution to Problem 14.1.60b into a single u-substitution. We now see that

$$\iint_R x^{2r-1}y^{s-1}f(x^ry^s)dA = \int_0^1\int_0^1 x^{2r-1}y^{s-1}f(x^ry^s)dydx \tag{23}$$

$$\overset{u=x^ry^s}{=} \int_0^1\int_0^1 x^{r-1}f(x^ry^s)x^ry^{s-1}dydx \overset{u=x^ry^s}{=} \int_0^1\int_{y=0}^1 \frac{x^{r-1}}{s}f(u)dudx \tag{24}$$

$$= \int_0^1 \left(\frac{x^{r-1}}{s}F(u)\Big|_{y=0}^1\right)dx = \int_0^1\left(\frac{x^{r-1}}{s}F(x^ry^s)\Big|_{y=0}^1\right)dx \tag{25}$$

$$= \int_0^1\left(\frac{x^{r-1}}{s}F(x^r) - \frac{x^{r-1}}{s}\underbrace{F(0)}_{=0}\right)dx = \int_0^1 \frac{x^{r-1}}{s}F(x^r)dx \tag{26}$$

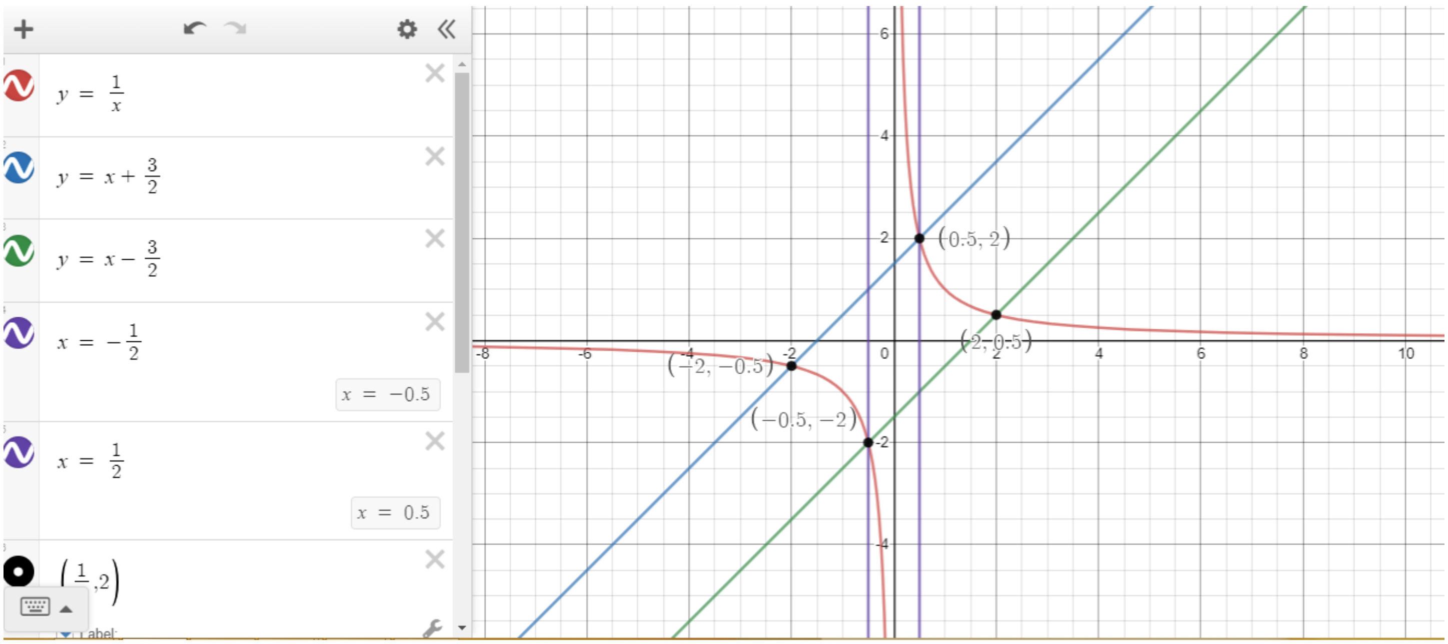
$$\overset{u=x^r}{=} \int_0^1 \frac{1}{rs}F(x^r)rx^{r-1}dx \overset{u=x^r}{=} \int_{x=0}^1 \frac{1}{rs}F(u)du \tag{27}$$

$$= \frac{1}{rs}G(u)\Big|_{x=0}^{1} = \frac{1}{rs}G(x^r)\Big|_{x=0}^{1} = \frac{G(1) - G(0)}{rs}. \tag{28}$$

Problem 4: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.
(a) Find the area of R.
(b) Evaluate

$$\iint_R xy\,dA. \tag{29}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x + \frac{3}{2} \end{matrix} \to \frac{1}{x} = x + \frac{3}{2} \to x^2 + \frac{3}{2}x - 1 = 0 \tag{30}$$

$$\to x = -2, \frac{1}{2} \to (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2). \tag{31}$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x - \frac{3}{2} \end{matrix} \to \frac{1}{x} = x - \frac{3}{2} \to x^2 - \frac{3}{2}x - 1 = 0 \tag{32}$$

$$\rightarrow x = -\frac{1}{2}, 2 \rightarrow (x, y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}). \tag{33}$$

We now see that the area of R is

$$\iint_R 1 dA = \iint_R 1 dy dx \tag{34}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1 dy dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1 dy dx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1 dy dx \tag{35}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(y \Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(y \Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(y \Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{36}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3 dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x + \frac{3}{2} \right) dx \tag{37}$$

$$\left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + 3x \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Big|_{\frac{1}{2}}^{2} \tag{38}$$

$$= (1 + 2\ln(2) - \frac{5}{8}) + 3 + (1 + 2\ln(2) - \frac{5}{8}) = \boxed{\frac{15}{4} + 4\ln(2)}. \tag{39}$$

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy dA = \iint_R xy dy dx \tag{40}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} xy dy dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xy dy dx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} xy dy dx \tag{41}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}}\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}}\right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}}\right) dx \tag{42}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(\frac{1}{x})^2\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}x(\frac{1}{x})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx \tag{43}$$

$$= \frac{1}{2}\int_{-2}^{-\frac{1}{2}} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x}\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2 dx + \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x^3 + 3x^2 - \frac{9}{4}x\right) dx \tag{44}$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x|\right)\Big|_{-2}^{-\frac{1}{2}} + x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2}\left(\ln|x| - \frac{1}{4}x^4 + x^3 - \frac{9}{8}x^2\right)\Big|_{\frac{1}{2}}^{2} \tag{45}$$

$$= \boxed{2\ln(2) - \frac{5}{64}} \tag{46}$$

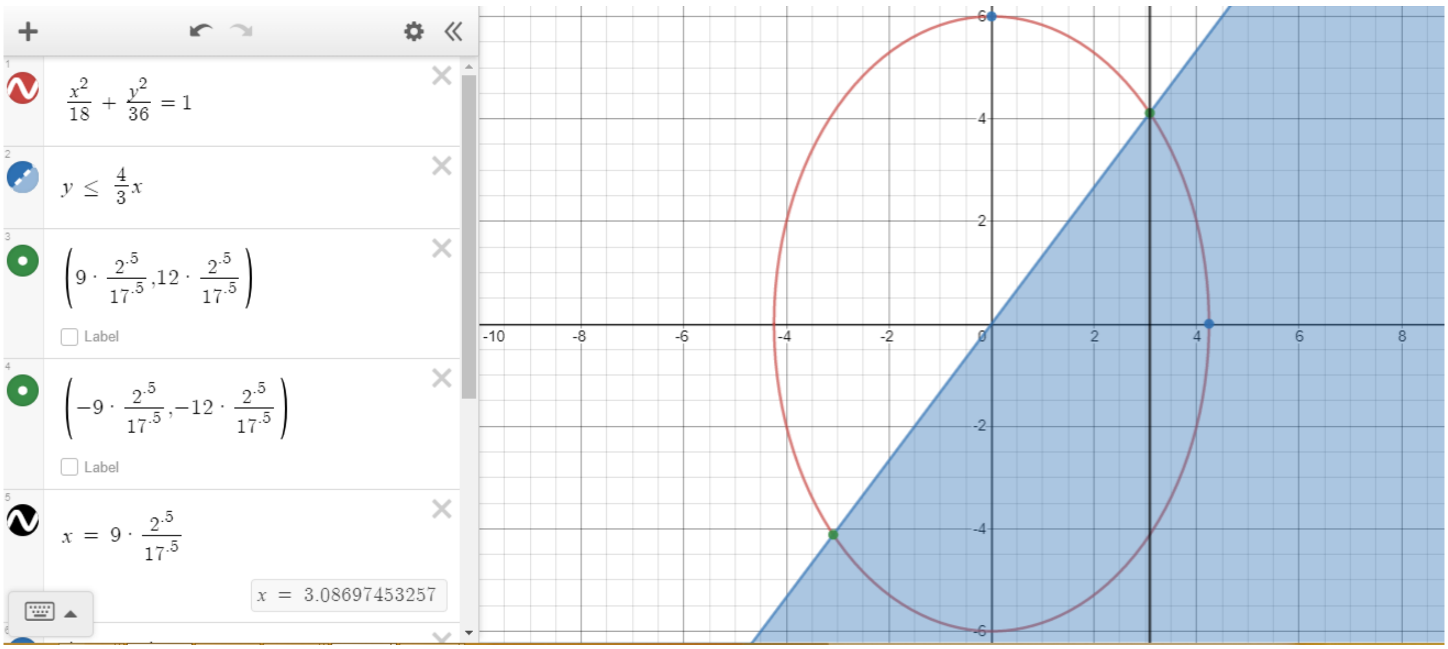
Problem 5: Let R be the region inside of the ellipse $\frac{x^2}{18}+\frac{y^2}{36}=1$ for which we also have $y \le \frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy\,dA. \tag{47}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18}+\frac{y^2}{36}=1$ and $y=\frac{4}{3}x$. We see that

$$\begin{matrix}\frac{x^2}{18}+\frac{y^2}{36}=1\\ y=\frac{4}{3}x\end{matrix} \rightarrow \frac{x^2}{18}+\frac{\frac{16}{9}x^2}{36}=1 \tag{48}$$

$$\rightarrow x=\pm\frac{9\sqrt{2}}{\sqrt{17}} \rightarrow (x,y)=\left(-\frac{9\sqrt{2}}{\sqrt{17}},-\frac{12\sqrt{2}}{\sqrt{17}}\right),\left(\frac{9\sqrt{2}}{\sqrt{17}},\frac{12\sqrt{2}}{\sqrt{17}}\right). \tag{49}$$

We now see that the area of R is

$$\iint_R 1\,dA=\iint_R 1\,dy\,dx \tag{50}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1 dy dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1 dy dx \tag{51}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{52}$$

. . . .

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36 - 2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36 - 2x^2} dx \tag{53}$$

Since

$$\int \sqrt{1 - x^2} = \frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}(x), \quad \text{(substitute } x = \sin(\theta)) \tag{54}$$

we see that

$$\int \sqrt{36 - 2x^2} dx = \int 6\sqrt{1 - (\frac{x}{3\sqrt{2}})^2} dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1 - y^2} dy \tag{55}$$

. . . .

$$= 9\sqrt{2}y\sqrt{1 - y^2} + 9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36 - 2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}). \tag{56}$$

Applying this result to equation (53), we see that

$$\int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36 - 2x^2}\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36 - 2x^2} dx \tag{57}$$

. . . .

$$= \left(\frac{2}{3}x^2 + \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Bigg|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} + \left(x\sqrt{36-2x^2} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Bigg|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{58}$$

. . . .

$$2\left(\frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Bigg|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{59}$$

. . . .

$$x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} - x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}} - 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \tag{60}$$

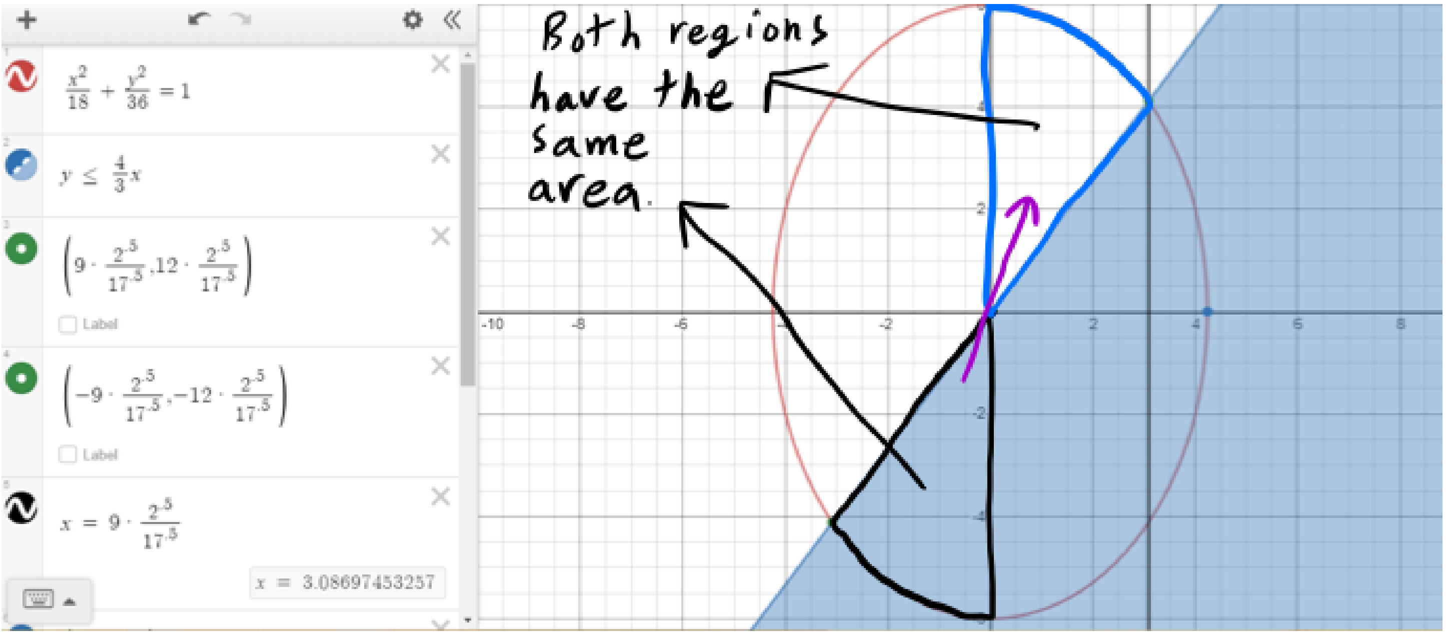
. . . .

$$= x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}} \tag{61}$$

. . . .

$$= 0 + 18\sqrt{2}\sin^{-1}(1) = \boxed{9\sqrt{2}\pi}. \tag{62}$$

Remark: For the ellipse $\frac{y^2}{36} + \frac{x^2}{18} = 1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6 \cdot 3\sqrt{2} \cdot \pi = 18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x, y) across the origin we get the point $(-x, -y)$, and that reflection across the origin (or reflection across any other point) preserves area as shown in the picture below.



Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy\,dA = \iint_R xy\,dy\,dx \tag{63}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy\,dy\,dx \tag{64}$$

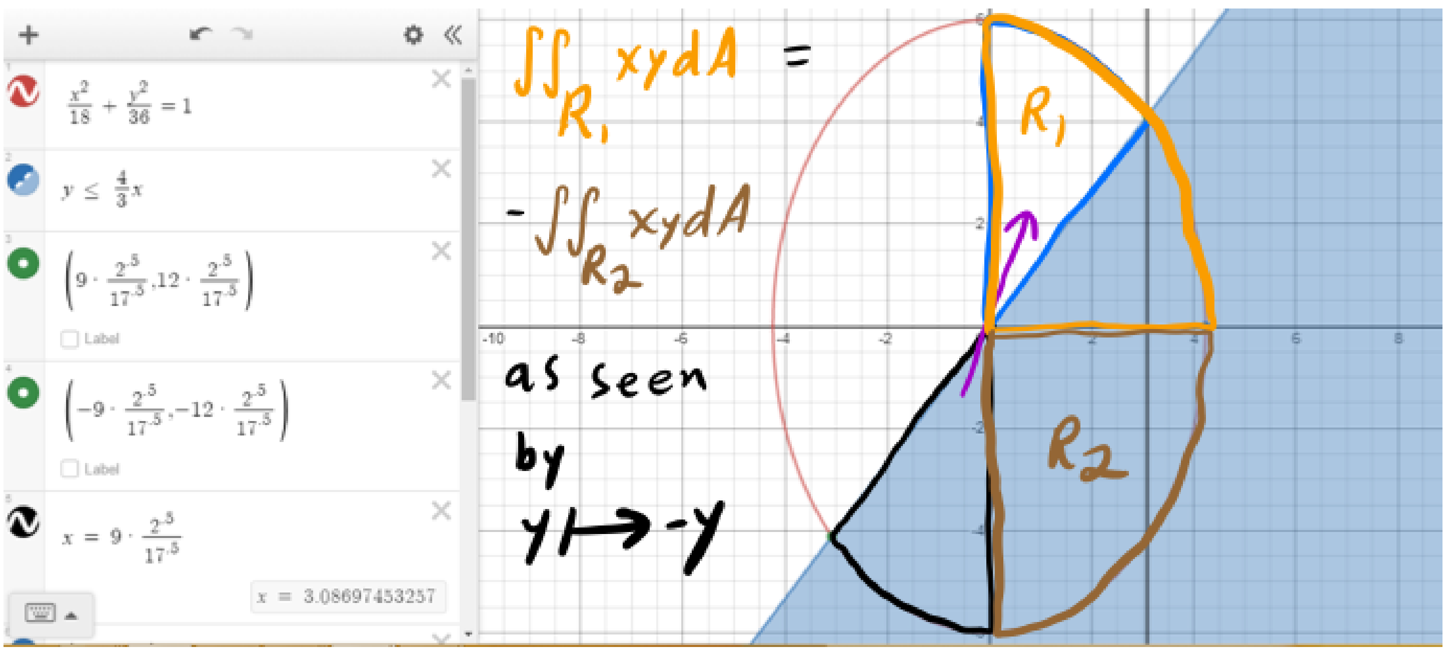
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{65}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \tag{66}$$

$$+ \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{67}$$

Remark: We see that both integrals appearing in equation (64) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0 as shown in the picture below.



Problem 6: Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

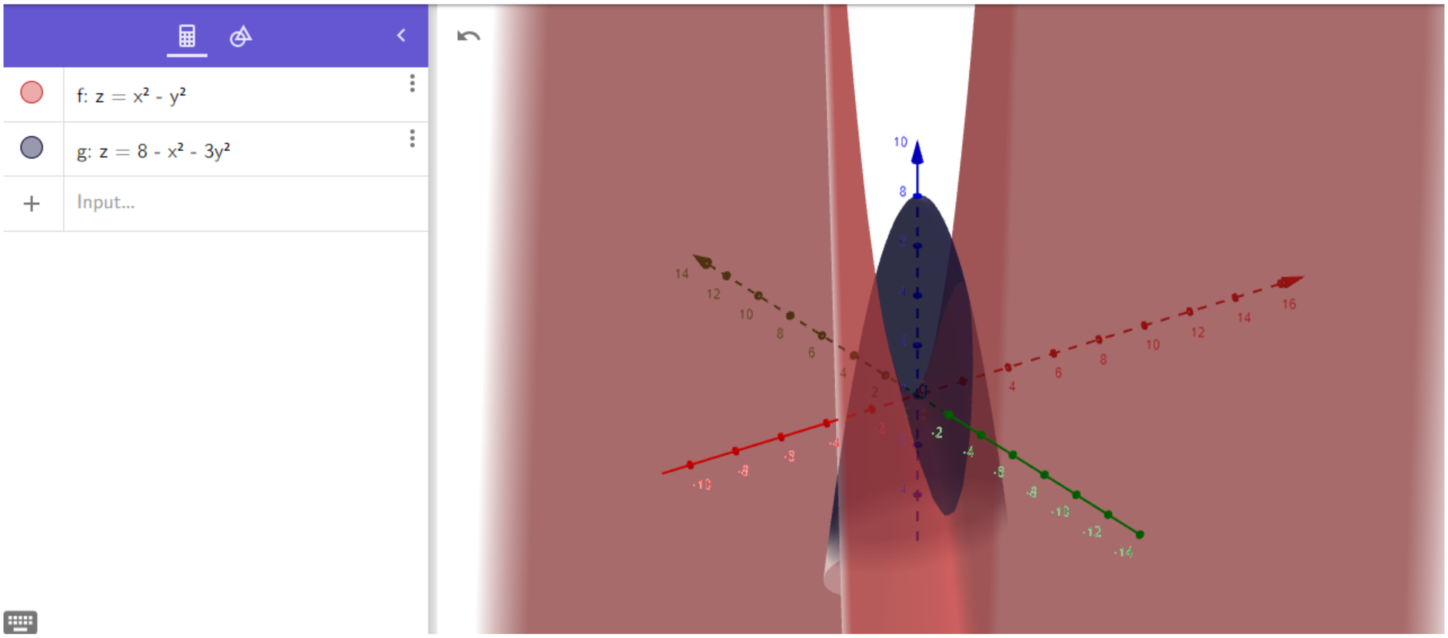


FIGURE 1. A view of the solid S whose volume we are calculating.

Solution: We begin by finding the (x, y)-coordinates of the curves of intersection of the 2 given surfaces. We see that

$$8 - x^2 - 3y^2 = z = x^2 - y^2 \to 2x^2 + 2y^2 = 8 \to x^2 + y^2 = 4, \tag{68}$$

so the (x, y)-coordinates of the curve of intersection is simply the circle of radius 2 centered at the origin.

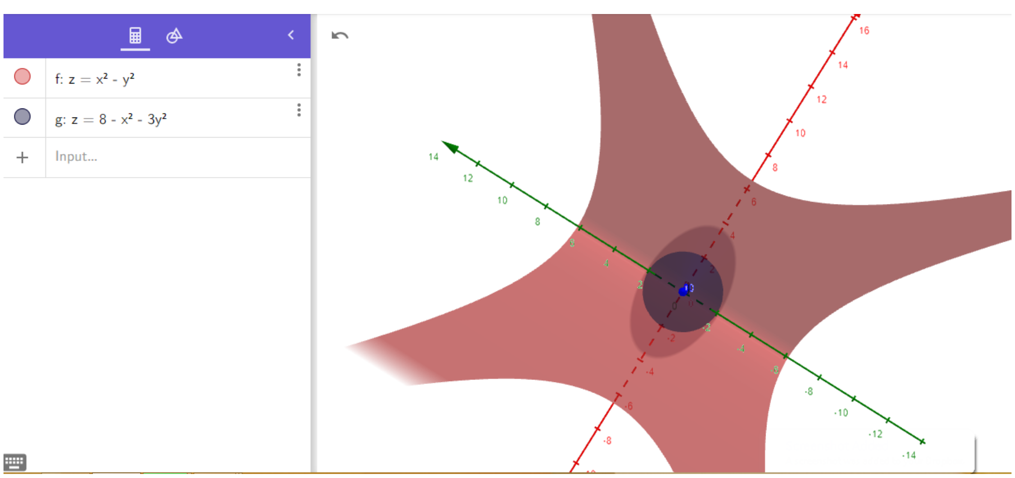


FIGURE 2. A bird's eye view of the solid S that is used to find the region of integration R.

Noting that $8-0^2-3\cdot 0^2 = 8 > 0 = 0^2-0^2$, we see that the curve $z = 8-x^2-3y^2$ lies above the curve $z = x^2-y^2$ for all (x,y) inside of R, the disc of radius 2 centered at the origin. We now see that

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R ((8-x^2-3y^2)-(x^2-y^2))dA \tag{69}$$

$$= \iint_R (8-2x^2-2y^2)dA = \int_0^{2\pi}\int_0^2 (8-2r^2)r\,dr\,d\theta \tag{70}$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r-2r^3)dr\right) = (2\pi)\left(4r^2 - \frac{1}{2}r^4\Big|_0^2\right) = \boxed{16\pi}. \tag{71}$$