Problem 1: Consider the function $f(x, y) = \ln(1 + 4x^2 + 3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

a. Find the gradient field $\nabla f(x, y)$ of $f(x, y)$ and then evaluate it at P.

b. Find the angles θ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.

c. Write the directional derivative at P as a function of θ; call this function $g(\theta)$.

d. Find the value of θ that maximizes $g(\theta)$ and find the maximum value.

e. Verify that the value of θ that maximizes g corresponds to the direction of the gradient vector at P. Verify that the maximum value of g equals the magnitude of the gradient vector at P.

Solution to a: We see that

$$\begin{aligned} f_x(x, y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial x}(1 + 4x^2 + 3y^2) = \tfrac{8x}{1+4x^2+3y^2} \\ f_y(x, y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial y}(1 + 4x^2 + 3y^2) = \tfrac{6y}{1+4x^2+3y^2} \end{aligned} \tag{1}$$

$$\to \nabla f(x, y) = \langle \frac{8x}{1 + 4x^2 + 3y^2}, \frac{6y}{1 + 4x^2 + 3y^2} \rangle. \tag{2}$$

$$\nabla f(\frac{3}{4}, -\sqrt{3}) = \langle \frac{6}{1 + \frac{9}{4} + 9}, \frac{-6\sqrt{3}}{1 + \frac{9}{4} + 9} \rangle = \langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle. \tag{3}$$

Solution to b: We recall that $\nabla f(P)$ points in the direction of maximum increase from P. Since $\nabla f(P)$ is in the fourth quadrant, we see that

$$\theta_{\text{max}} = \tan^{-1}(\frac{\frac{-24\sqrt{3}}{49}}{\frac{24}{49}}) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}. \tag{4}$$

is the angle associated with the direction of maximum increase. Since $-\nabla f(P)$ points in the direction of maximum decrease from P, we see that $\theta_{\text{min}} = \theta_{\text{max}} + \pi = \frac{2\pi}{3}$ is the angle associated with the direction of maximum decrease. Since the directions of no change are orthogonal to $\nabla f(P)$ (and to $-\nabla f(P)$), we see that $\theta_1 = \theta_{\max} + \frac{\pi}{2} = \frac{5\pi}{6}$ and $\theta_2 = \theta_{\max} - \frac{\pi}{2} = -\frac{\pi}{6}$ are the angles associated to the directions of zero change.

Solution to c: We recall that $\vec{u}(\theta) = \langle \cos(\theta), \sin(\theta)\rangle$ is the unit vector associated with the angle θ. We also recall that for any unit vector $\vec{u}$, we have that

$$d_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}, \text{ so} \tag{5}$$

$$g(\theta) = d_{\vec{u}(\theta)} f(P) = \nabla f(P) \cdot \vec{u}(\theta) = \langle \frac{24}{49}, \frac{-24\sqrt{3}}{49}\rangle \cdot \langle \cos(\theta), \sin(\theta)\rangle \tag{6}$$

$$= \frac{24}{49}\cos(\theta) - \frac{24\sqrt{3}}{49}\sin(\theta). \tag{7}$$

Solution to d: We see that

$$g'(\theta) = -\frac{24}{49}\sin(\theta) - \frac{24\sqrt{3}}{49}\cos(\theta) \to \tag{8}$$

$$g'(\theta) = 0 \Leftrightarrow -\frac{24}{49}\sin(\theta) = \frac{24\sqrt{3}}{49}\cos(\theta) \Leftrightarrow \tan(\theta) = -\sqrt{3} \Leftrightarrow \tag{9}$$

$$\theta = -\frac{\pi}{3}, \frac{2\pi}{3} \tag{10}$$

We see that

$$g''(\theta) = -\frac{24}{49}\cos(\theta) + \frac{24\sqrt{3}}{49}\sin(\theta) \tag{11}$$

$$\to g''(-\frac{\pi}{3}) = -\frac{24}{49}\cos(-\frac{\pi}{3}) + \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = -\frac{48}{89} < 0. \tag{12}$$

The second derivative test shows us that $g(\theta)$ has a local maximum at $\theta = -\frac{\pi}{3}$.

$$g(-\frac{\pi}{3}) = \frac{24}{49}\cos(-\frac{\pi}{3}) - \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = \frac{48}{49}. \tag{13}$$

we see that g attains its maximum value of $\frac{48}{89}$ on $[0, 2\pi]$ at $\theta = -\frac{\pi}{3}$.

Solution to e: From parts b and d we have already seen that the value of θ that maximizes $g(\theta)$ is the same as the angle θ associated with the direction of maximum increase. To finish, we just note that

$$|\nabla f(\frac{3}{4}, -\sqrt{3})| = |\langle\frac{24}{49}, \frac{-24\sqrt{3}}{49}\rangle| = \frac{24}{49}|\langle 1, -\sqrt{3}\rangle| \tag{14}$$

$$= \frac{24}{49}\sqrt{1^2 + (-\sqrt{3})^2} = \frac{48}{49}. \tag{15}$$

Problem 2: Consider the function $f(x,y) = x^2 + y^2$ and and the point $P = (2,3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.

Solution to (a): We see that $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2x, 2y\rangle$. We see that $-\nabla f(2,3) = \langle -4, -6\rangle$ is a vector that points in the direction of maximum decrease of f at the point P. Since $|\langle -4, -6\rangle| = \sqrt{52} = 2\sqrt{13}$, we see that

$$\frac{\langle -4, -6\rangle}{|\langle -4, -6\rangle|} = \frac{1}{2\sqrt{13}}\langle -4, -6\rangle = \boxed{\langle \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\rangle} \tag{16}$$

is the direction of maximum decrease of f at the point P.

Solution to (b): We see that $|\vec{u}| = \sqrt{13}$, so

$$\vec{w} = \frac{\vec{u}}{|\vec{u}|} = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \tag{17}$$

is the unit vector that points in the same direction as $\vec{u}$, so

$$d_{\vec{w}} f(2,3) = \nabla f(2,3) \cdot \vec{w} = \langle 4, 6\rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle = \boxed{\frac{24}{\sqrt{13}}}. \tag{18}$$

Problem 3: Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.

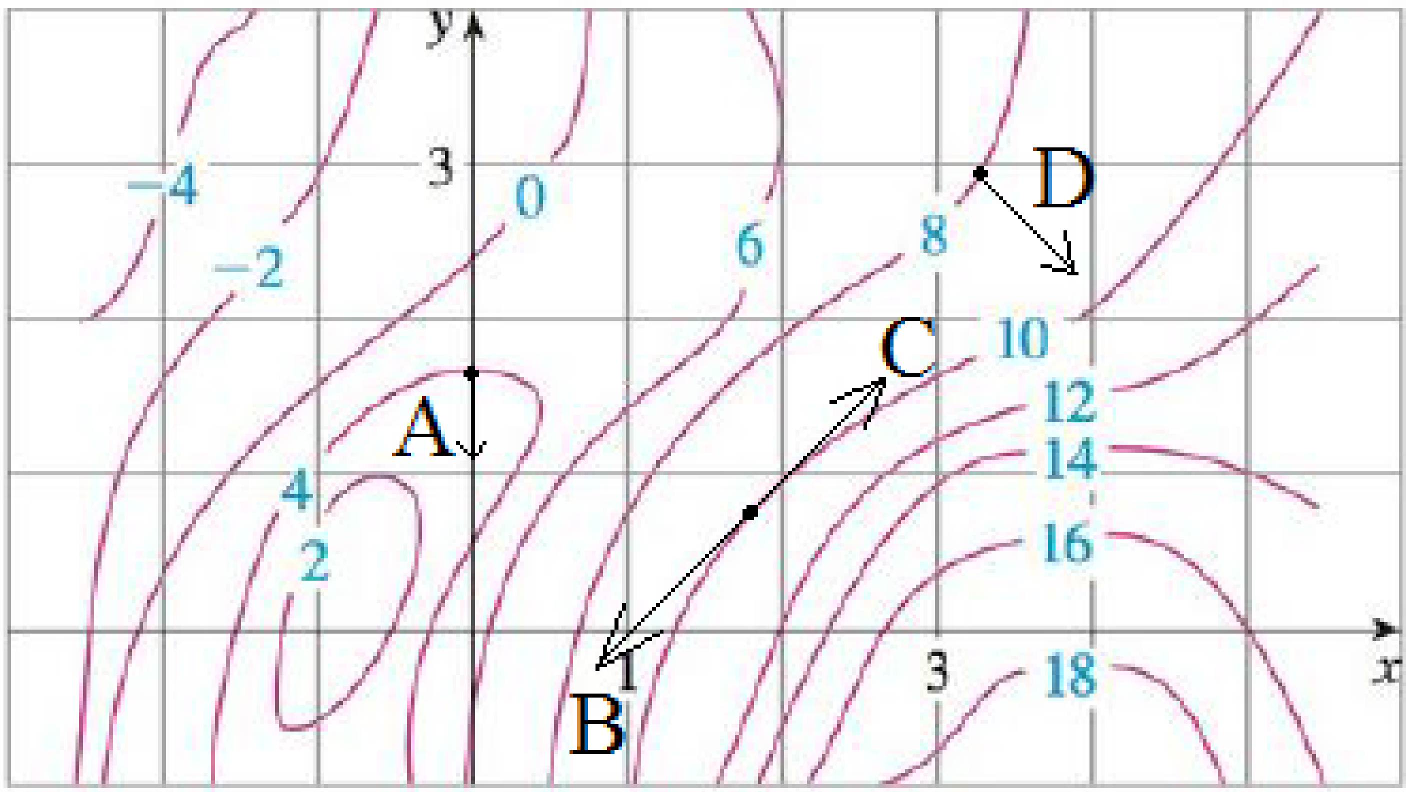

FIGURE 1. Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) $\boxed{(D)}$

None of the vectors could possibly be a gradient vector for $f(x, y)$.

Explanation: The gradient vector of a function $f(x, y)$ is normal to the level curves (the curves of the form $f(x, y) = c$, with c a constant) and points in the direction of maximum increase. We see that vector A is normal to a level curve of f, but points in the direction of decrease and is therefore not a gradient vector. We see that vectors B and C are tangent to a level curve, not normal to the level curve, so neither of them can be a gradient vector. We see that vector D is normal to a level curve of f and points in the direction of increase, so D could be a gradient vector of f.

Problem 4: Determine all critical points of the function $f(x,y) = x^3 - y^3 + xy$, then classify each of the critical points as a local maximum, local minimum, or saddle point.

Solution: To find the criticial points of f, we simply have to find all (x,y) for which both partial derivatives of f are 0.

$$\begin{array}{rcl} f_x(x,y) &=& 0 \\ f_y(x,y) &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} 3x^2 + y &=& 0 \\ -3y^2 + x &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} -3x^2 &=& y \\ 3y^2 &=& x \end{array} \tag{19}$$

$$\to x = 3(-3x^2)^2 = 27x^4 \to x = 0, \frac{1}{3} \to (x,y) = \boxed{(0,0), (\frac{1}{3}, -\frac{1}{3})}. \tag{20}$$

We now proceed to calculate all of the second derivatives of f as well as the discriminant function so that we can apply the second derivative test.

$$\begin{array}{rcl} f_{xx}(x,y) &=& 6x \\ f_{yy}(x,y) &=& -6y \\ f_{xy}(x,y) &=& 1 \end{array} \tag{21}$$

$$\to D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = -36xy - 1. \tag{22}$$

Since $D(0,0) = -1 < 0$, we see that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $D(\frac{1}{3}, -\frac{1}{3}) = 3 > 0$ and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 2 > 0$ we see that $\boxed{(\frac{1}{3}, -\frac{1}{3}) \text{ is a local minimum}}$.

Problem 5: Show that the second derivative test is inconclusive when applied to the function $f(x,y) = x^4y^2$ at the point $(0,0)$. Show that $f(x,y)$ has a local minimum at $(0,0)$ by direct analysis.
Hint: The product of 2 negative numbers is positive.

Solution: We will first verify that $(0,0)$ is a critical point. We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3y^2 \text{ and } \frac{\partial f}{\partial y}(x,y) = 2x^4y, \text{ so} \tag{23}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3y^2 = 0 \\ 2x^4y = 0 \end{matrix} \Leftrightarrow x = 0 \text{ or } y = 0. \tag{24}$$

It follows that the critical points of f are precisely those points which are on either the x-axis or the y-axis, and $(0,0)$ is certainly such a point. Next, we notice that

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3y^2) = 12x^2y^2, \tag{25}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(2x^4y) = 2x^4, \text{ and} \tag{26}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(2x^4y) = 8x^3y, \text{ so} \tag{27}$$

$$\begin{aligned} D(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 \\ &= 12x^2y^2 \cdot 2x^4 - (8x^3y)^2 = -40x^6y^2. \end{aligned} \tag{28}$$

Since $D(x,y) = 0$ whenever $x = 0$ or $y = 0$, we see that the second derivative test is inconclusive for every critical point of f (which includes (0,0)). However, we are still able to describe the behavior of $f(x,y)$ at any of its critical points by using a direct analysis. Note that $x^4y^2 \geq 0$ for all $(x,y) \in \mathbb{R}^2$ (use the hint if this is not obvious to you), and that $x^4y^2 = 0$ whenever $x = 0$ or $y = 0$. It follows that f attains its absolute minimum at any of its critical points.

Problem 6: Consider the function $f(x,y) = 3+x^4+3y^4$. Show that $(0,0)$ is a critical point for $f(x,y)$ and show that the second derivative test is inconclusive at $(0,0)$. Then describe the behavior of $f(x,y)$ at $(0,0)$.

Solution: We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \text{ and } \frac{\partial f}{\partial y}(x,y) = 12y^3, \text{ so} \tag{29}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3 = 0 \\ 12y^3 = 0 \end{matrix} \Leftrightarrow (x,y) = (0,0). \tag{30}$$

It follows that $(0,0)$ is the only critical point of f in all of $\mathbb{R}^2$. We also note that

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3) = 12x^2, \tag{31}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(12y^3) = 36y^2, \text{ and} \tag{32}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(12y^3) = 0, \text{ so} \tag{33}$$

$$D(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 = 12x^2 \cdot 36y^2 - 0^2 = 432x^2y^2. \tag{34}$$

Since $D(0,0) = 0$, we see that the second derivative test is inconclusive. However, we are still able to describe the behavior of $f(x,y)$ at $(0,0)$. Note that $x^4 \geq 0$ for all $x \in \mathbb{R}$, and $3y^4 \geq 0$ for all $y \in \mathbb{R}$. Furthermore, $x^4 = 0$ if and only if $x = 0$, and $3y^4 = 0$ if and only if $y = 0$. It follows that $x^4 + 3y^4 \geq 0$ for all $(x,y) \in \mathbb{R}^2$, and $x^4 + 3y^4 = 0$ if and only if $(x,y) = (0,0)$. From this we are able to see that $f(x,y) = 3 + x^4 + 3y^4$ attains an absolute minimum at $(0,0)$.