Problem 15.2.5.82: Verify that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = 1. \tag{1}$$

Solution: We begin by reviewing one of the sum to product trigonometric identities. Observe that

$$\boxed{\sin(x)+\sin(y)} = \sin(\frac{x+y}{2}+\frac{x-y}{2}) + \sin(\frac{x+y}{2}-\frac{x-y}{2}) \tag{2}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(-\frac{x-y}{2}) + \sin(-\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{3}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(+\frac{x-y}{2}) - \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{4}$$

$$= \boxed{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}. \tag{5}$$

Recalling that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \tag{6}$$

we see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = \lim_{(x,y)\to(0,0)} \frac{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}{x+y} \tag{7}$$

$$\left(\lim_{(x,y)\to(0,0)} \frac{\sin(\frac{x+y}{2})}{\frac{x+y}{2}}\right)\left(\lim_{(x,y)\to(0,0)} \cos(\frac{x-y}{2})\right) = (1)(\cos(\frac{0-0}{2})) = 1. \tag{8}$$

Homemade Problem: Consider the function

$$f(x,y) = \frac{xy^2}{x^2+y^4}. \tag{9}$$

(a) Show that if L is a line that passes through the origin, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = 0. \tag{10}$$

(b) Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \tag{11}$$

does not exist.

Solution to (a): Firstly, we see that if L is a line of the form $y = mx$ for some $m \in \mathbb{R}$, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0}\frac{x(mx)^2}{x^2+(mx)^4} \tag{12}$$

$$= \lim_{x\to 0}\frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to 0}\frac{m^2x}{1+m^4x^2} = 0. \tag{13}$$

The only line L left to consider is the line through the origin with infinite slope, which is just the line $x = 0$. In this case we see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0}\frac{0\cdot y^2}{0^2+y^4} = 0. \tag{14}$$

Solution to (b): In order to show that the limit in equation (11) does not exist we need to use the 2 path test. Based on part (a), we see that our second path needs cannot be a line. Thankfully, we only need to find a path P that results in any nonzero value when the limit is taken along P. If we try the parabolic path $y = x^2$, then we again get a value of 0 for the limit, but if we try the path $x = y^2$ then we get a value of $\frac{1}{2}$! In fact, we see that for $m \in \mathbb{R}$ and the path P_m given by $x = my^2$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f(x,y) = \lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{(my^2)y^2}{(my^2)^2 + y^4} \tag{15}$$

$$= \lim_{y\to 0} \frac{my^4}{m^2y^4 + y^4} = \lim_{y\to 0} \frac{m}{m^2+1}. \tag{16}$$

Since the range of the function $g(m) = \frac{m}{m^2+1}$ is $[-1, 1]$, we see that the limit can take on any value between -1 and 1 if the correct path is chosen. While we only need 2 paths that result in different values to apply the 2 path test, it is amusing to see that we have found infinitely many paths that result in infinitely many different values.

Problem 15.3.97: Consider the function $f(x, y) = \sqrt{|xy|}$.

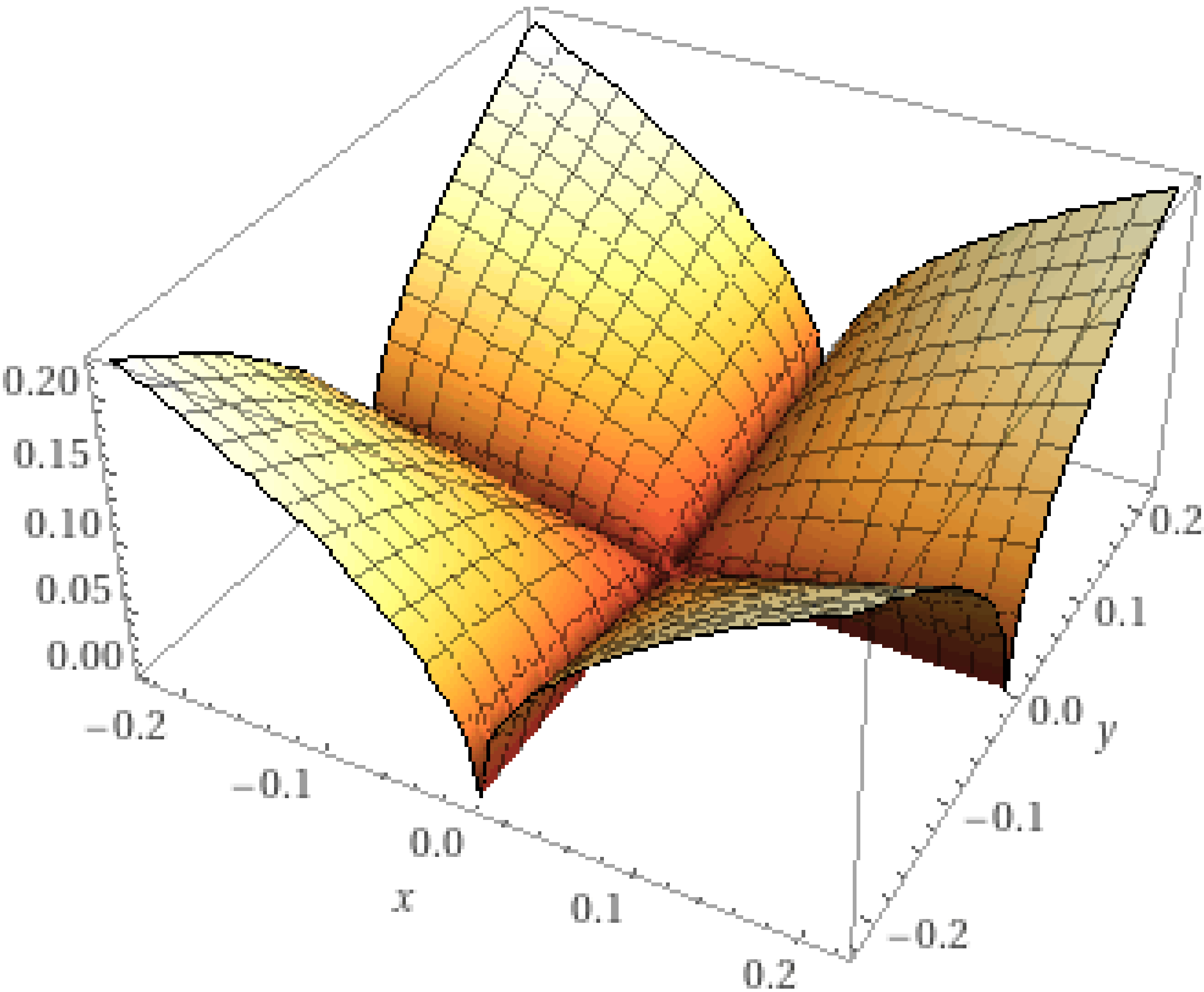

FIGURE 1. A graph of $z = \sqrt{|xy|}$.

(a) Is f continuous at $(0, 0)$?
(b) Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist by calculating their values.
(c) Determine whether f_x and f_y are continuous at $(0, 0)$.
(d) Is f differentiable at $(0, 0)$?

Solution to (a): Yes. We will show that $f(x, y)$ is continuous on all of $\mathbb{R}^2$. The function $f_1(x, y) = xy$ is a continuous function since it is a polynomial function. The function $f_2(x) = |x|$ is also a continuous function, and the composition of continuous functions is again continuous, so we see that $f_3(x, y) := f_2(f_1(x, y)) = |xy|$ is a continuous function. Since $|xy|$ only takes on nonnegative values and the function $f_4(x) = \sqrt{x}$ is continuous on the domain $[0, \infty)$, we see that $f(x, y) = f_4(f_3(x, y))$ is indeed a continuous function.

Solution to (b): We see that

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{\sqrt{|0\cdot y|} - \sqrt{|0\cdot 0|}}{y} = 0, \text{ and} \tag{17}$$

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt{|x\cdot 0|} - \sqrt{|0\cdot 0|}}{x} = 0. \tag{18}$$

Solution to (c): We will show that neither of f_x and f_y are continuous at $(0, 0)$. We note that for all $x, y > 0$ we have $f(x, y) = \sqrt{|xy|} = \sqrt{xy}$. It follows that for $x, y > 0$ we have

$$f_x(x,y) = \frac{\partial}{\partial x}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot y = \frac{1}{2}\sqrt{\frac{y}{x}}, \text{ and} \tag{19}$$

$$f_y(x,y) = \frac{\partial}{\partial y}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}. \tag{20}$$

We now use the 2 path test to show that neither function is continuous. Let us consider the path P_m given by $y = mx$ with $m, x > 0$ so that the path lies in the first quadrant. We see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_x(x,y) = \lim_{x\to 0^+} f_x(x,mx) = \lim_{x\to 0^+} \frac{1}{2}\sqrt{\frac{mx}{x}} = \frac{m}{2}, \text{ and} \tag{21}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_y(x,y) = \lim_{x\to 0^+} f_y(x,mx) = \lim_{x\to 0^+} \frac{1}{2}\sqrt{\frac{x}{mx}} = \frac{1}{2m}. \tag{22}$$

We see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and 1 respectively for the value of $f_x(x, y)$ as (x, y) approaches $(0, 0)$, so f_x is not continuous at $(0, 0)$. Similarly, we see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and $\frac{1}{4}$ respectively for the value of $f_y(x, y)$ as (x, y) approaches $(0, 0)$, so f_y is not continuous at $(0, 0)$.

Solution to (d): No. We begin by examining the directional derivative in the direction of the vector $\hat{u} = \langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ at $(0,0)$. We see that

$$D_{\hat{u}}f(0,0) = \lim_{t\to 0}\frac{f\left((0,0)+t\hat{u}\right) - f(0,0)}{t} = \lim_{t\to 0}\frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - 0}{t} \tag{23}$$

$$= \lim_{t\to 0}\frac{\sqrt{|\frac{t^2}{2}|}}{t} = \lim_{t\to 0}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{24}$$

If f was differentiable at $(0,0)$, then we would have

$$D_{\hat{u}}f(0,0) = \nabla f(0,0)\cdot\hat{u} = \langle f_x(0,0), f_y(0,0)\rangle \cdot \langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = 0. \tag{25}$$

Since this is not the case, we see that f is not differentiable at $(0,0)$.