Problem 2: Let $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$.

a. Evaluate $\iint_R \cos(x\sqrt{y})dA$.

b. Evaluate $\iint_R x^3 y \cos(x^2 y^2)dA$.

Hint: Choose a convenient order of integration.

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a) i) $$\int_0^1 \underbrace{\int_0^1 \cos(x\sqrt{y})\,dx}\,dy$$

$$\approx \int_0^1 \cos(5x)\,dx$$

ii) $$\cancel{\int_0^1 \underbrace{\int_0^1 \cos(x\sqrt{y})\,dy}\,dx}$$

$$\cancel{\approx \int_0^1 \cos(2\sqrt{y})\,dy}$$

$$\int_0^1 \int_0^1 \cos(x\sqrt{y})\,dx\,dy$$

$$u = x\sqrt{y}, \quad du = \frac{\partial}{\partial x}(x\sqrt{y}) = \sqrt{y}\,dx$$

$$= \int_0^1 \int_{x=0}^1 \cos(u) \frac{du}{\sqrt{y}}\,dy$$

$$= \int_0^1 \left(\int_{x=0}^1 \cos(u)\,du\right) \frac{dy}{\sqrt{y}}$$

$$= \int_0^1 \left(\sin(u) \Big|_{x=0}^{1} \right) \frac{dy}{\sqrt{y}}$$

$$= \int_0^1 \left(\sin(x\sqrt{y}) \Big|_{x=0}^{1} \right) \frac{dy}{\sqrt{y}}$$

$\sin(0)=0$

$$= \int_0^1 \sin(\sqrt{y}) \frac{dy}{\sqrt{y}}$$

$v = \sqrt{y}$

$dv = \frac{1}{2\sqrt{y}} dy$

$\rightarrow 2dv = \frac{dy}{\sqrt{y}}$

$$= \int_{y=0}^{1} 2\sin(v)\, dv$$

$$= -2\cos(v) \Big|_{y=0}^{1} = -2\cos(\sqrt{y}) \Big|_0^1$$

$$= -2\cos(1) + 2\cos(0)$$

$$= \boxed{2 - 2\cos(1)}$$

b) $\iint_R x^3 y \cos(x^2 y^2)\, dA$

i) $\cancel{\int_0^1 \int_0^1 x^3 y \cos(x^2 y^2)\, dx\, dy}$

$\approx \int_0^1 2x^3 \cos(4x^2)\, dx \qquad (y=2)$

ii) $\int_0^1 \int_0^1 x^3 y \cos(x^2 y^2)\, dy\, dx$

$\int_0^1 8y \cos(4y^2)\, dy \qquad (x=2)$

$\int_0^1 \int_0^1 x^3 y \cos(x^2 y^2)\, dy\, dx \qquad u = x^2 y^2$

$du = \frac{\partial}{\partial y}(x^2 y^2)$

$= \frac{1}{2} \int_0^1 \int_0^1 x \cos(x^2 y^2)\, 2x^2 y\, dy\, dx \qquad = 2x^2 y\, dy$

$= \frac{1}{2} \int_0^1 x \left(\int_0^1 \cos(x^2 y^2)\, 2x^2 y\, dy \right) dx$

$$= \frac{1}{2}\int_0^1 x\left(\int_{y=0}^{1} \cos(u)\,du\right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(\sin(u)\Big|_{y=0}^{1} \right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(\sin(x^2y^2)\Big|_{y=0}^{1} \right)dx$$

$$= \frac{1}{2}\int_0^1 x\left(\sin(x^2) - \sin(0) \right)dx$$

$$= \frac{1}{2}\int_0^1 x\sin(x^2)\,dx$$

$v = x^2$

$dv = 2x\,dx$

$$= \frac{1}{4}\int_0^1 \sin(x^2)\,2x\,dx$$

$$= \frac{1}{4}\int_{x=0}^{1} \sin(v)\,dv = \frac{1}{4}\left(-\cos(v)\Big|_{x=0}^{1}\right)$$

$$= -\frac{1}{4}\cos(x^2)\Big|_0^1 = -\frac{1}{4}(\cos(1)-\cos(0))$$

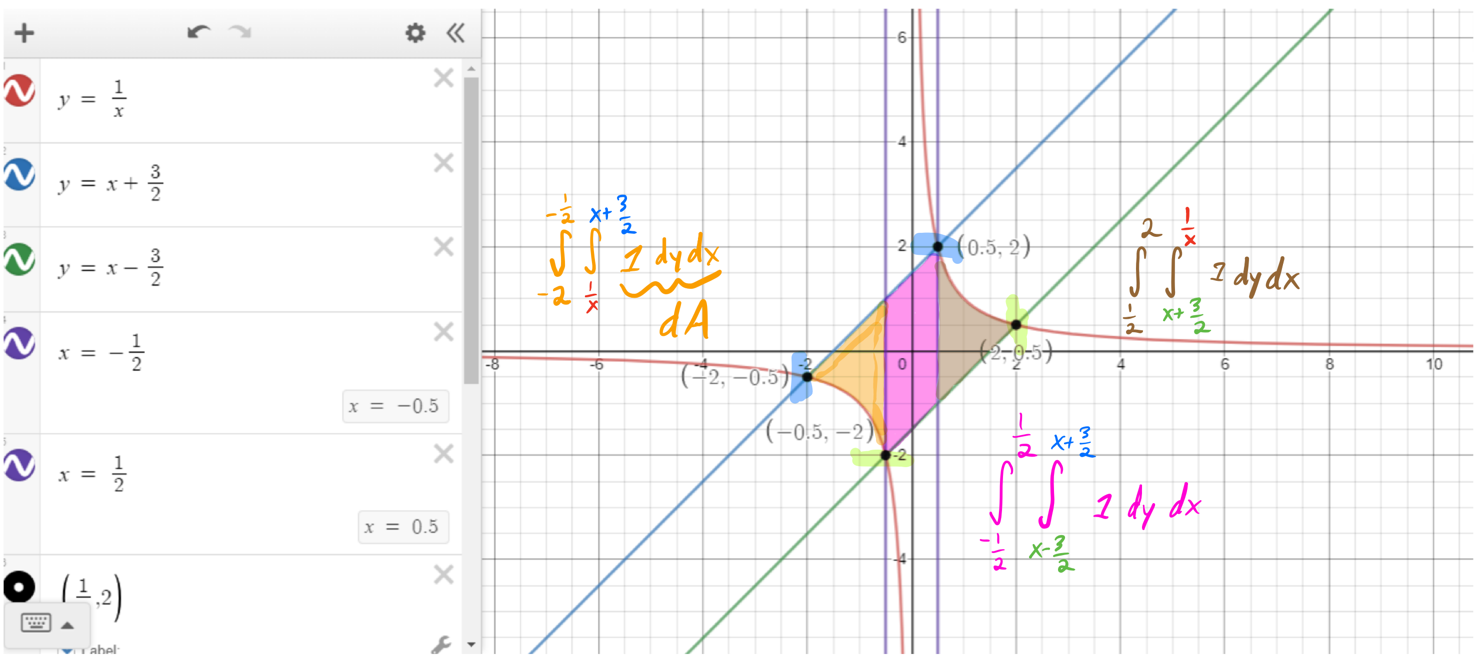
$$= \frac{1-\cos(1)}{4}$$

Problem 4: Let R be the region that is bounded by both branches of $y=\frac{1}{x}$, the line $y=x+\frac{3}{2}$, and the line $y=x-\frac{3}{2}$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy\,dA. \tag{3}$$



We solve

$y=\frac{1}{x}$, $y=x+\frac{3}{2}$ $\to$ $x+\frac{3}{2}=\frac{1}{x} \to x^2+\frac{3}{2}x=1$

$\to x^2+\frac{3}{2}x-1=0$

$$\to x=\frac{-\frac{3}{2}\pm\sqrt{\left(\frac{3}{2}\right)^2-4\cdot1\cdot(-1)}}{2\cdot1}$$

$$=\frac{-\frac{3}{2}\pm\sqrt{\frac{25}{4}}}{2}=\frac{-\frac{3}{2}\pm\frac{5}{2}}{2}=-2,\ \frac{1}{2}.$$

$y=\frac{1}{x} \to (x,y)=(-2,-\frac{1}{2}),(\frac{1}{2},2)$

We solve

$$y = \frac{1}{x}, \quad y = x - \frac{3}{2} \rightarrow \begin{matrix}\text{work}\\ \text{work}\\ \text{work}\end{matrix} \rightarrow (x,y) = \left(-\frac{1}{2}, -2\right), \left(2, \frac{1}{2}\right)$$

$$\text{Area} = \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1 \, dy\, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1 \, dy\, dx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1 \, dy\, dx$$

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For part b we just replace 1 with xy.

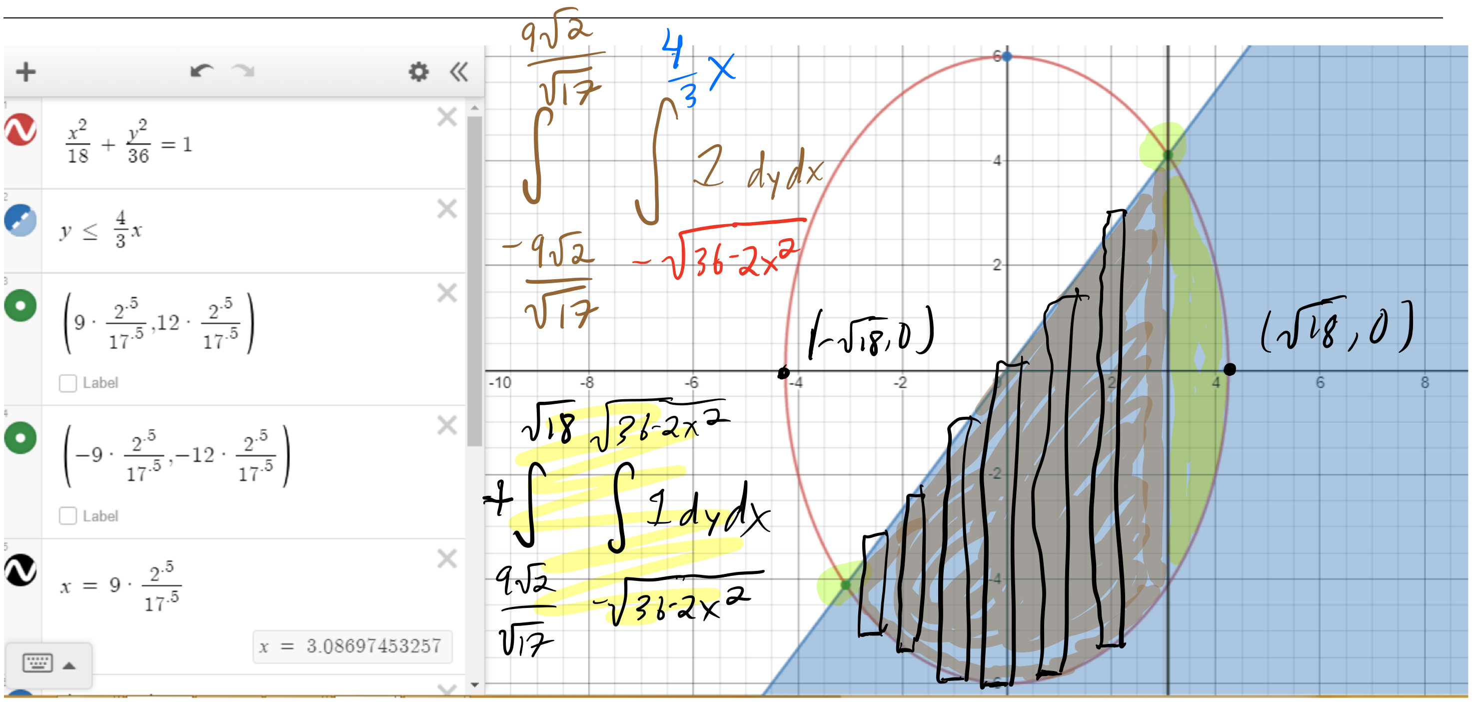
Problem 5: Let R be the region inside of the ellipse $\frac{x^2}{18} + \frac{y^2}{36} = 1$ for which we also have $y \leq \frac{4}{3}x$.

(a) Find the area of R.

$$\iint \underbrace{1\,dy\,dx}_{dA}$$

(b) Evaluate

$$\iint_R xy\,dA. \tag{4}$$



We solve

$$\frac{x^2}{18} + \frac{y^2}{36} = 1 \qquad y = \frac{4}{3}x \rightarrow \frac{x^2}{18} + \frac{(\frac{4}{3}x)^2}{36} = 1$$

$$\downarrow$$

$$\frac{x^2}{18} + \frac{\frac{16}{9}x^2}{36} = 1$$

$$2x^2 + \frac{16}{9}x^2 = 36 \rightarrow x^2 + \frac{8}{9}x^2 = 18$$

$\rightarrow 9x^2+8x^2=162 \rightarrow 17x^2=162$

$\rightarrow \quad x=\pm\sqrt{\frac{162}{17}}=\pm\frac{9\sqrt{2}}{\sqrt{17}}$

$y=\frac{4}{3}x \rightarrow (x,y)=\left(\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}\right),$

$\left(-\frac{9\sqrt{2}}{\sqrt{17}}, -\frac{12\sqrt{2}}{\sqrt{17}}\right)$

$\frac{x^2}{18}+\frac{y^2}{36}=1 \rightarrow 2x^2+y^2=36 \rightarrow y^2=36-2x^2$

$\rightarrow \quad y=\pm\sqrt{36-2x^2}$

(ellipse) $y_{top}=\sqrt{36-2x^2}$

(ellipse) $y_{bot}=-\sqrt{36-2x^2}$

Problem 6: Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

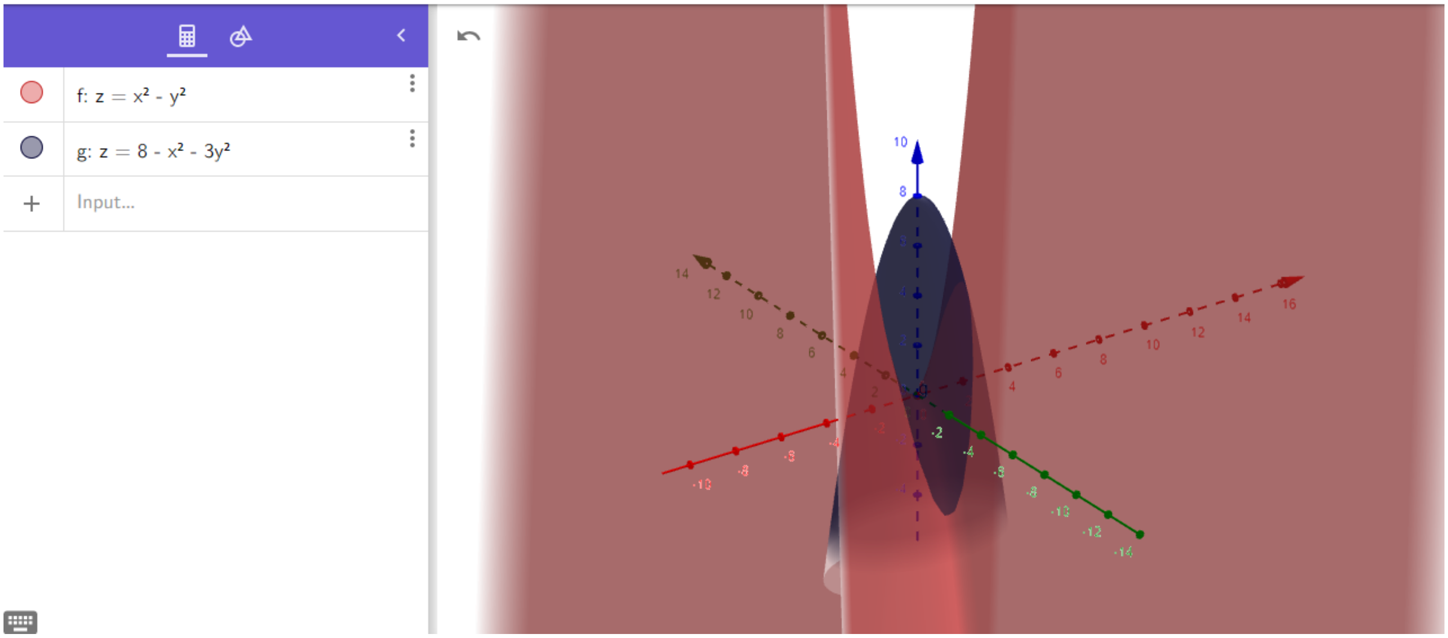
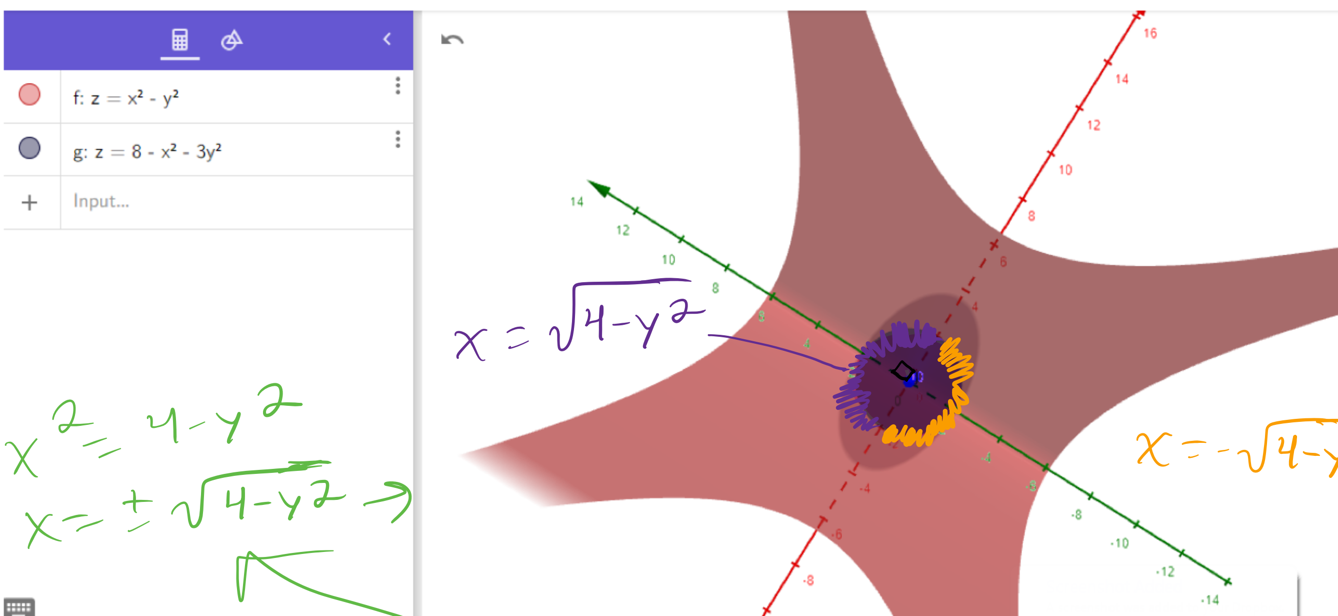


FIGURE 1. A view of the solid S whose volume we are calculating.



$x = \sqrt{4-y^2}$

$x = -\sqrt{4-y^2}$

$x^2 = 4 - y^2$

$x = \pm\sqrt{4-y^2} \rightarrow$

FIGURE 2. A bird's eye view of the solid S that is used to find the region of integration R.

$z = x^2 - y^2$

$z = 8 - x^2 - 3y^2$

$\rightarrow x^2 - y^2 = 8 - x^2 - 3y^2$

$\downarrow$

$2x^2 + 2y^2 = 8$

$\downarrow$

$x^2 + y^2 = 4 \rightarrow$

The (x,y)-coordinates of the intersection form a circle even though the intersection itself is not a circle.

$$\text{Volume} = \iint_{R} (z_{top} - z_{bot})\, dA$$

$R \leftarrow$ Shadow of intersection

$$R = \{(x,y) \mid x^2+y^2 \le 4\}$$

$$z_{top} = 8 - x^2 - 3y^2$$

$$z_{bot} = x^2 - y^2$$

$$z_{top} - z_{bot} = 8 - 2x^2 - 2y^2$$

$$\underbrace{\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (8 - 2x^2 - 2y^2)\, dx}_{\text{Pretend } y=2}\, dy$$

$$= \int_{-2}^{2} 8x - \frac{2}{3}x^3 - 2y^2 x \Big|_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}}$$

$$= \int_{-2}^{2} \Big[\left(8\sqrt{4-y^2} - \tfrac{2}{3}\left(\sqrt{4-y^2}\right)^3 - 2y^2\sqrt{4-y^2}\right)$$

$$- \left(-8\sqrt{4-y^2} - \tfrac{2}{3}\left(-\sqrt{4-y^2}\right)^3 + 2y^2\sqrt{4-y^2}\right) \Big] dy$$

$$= \int_{-2}^{2} \left(16\sqrt{4-y^2} - \tfrac{4}{3}\left(4-y^2\right)^{\frac{3}{2}} - 4y^2\sqrt{4-y^2}\right) dy$$

$x = 8 - z^2 - 3y^2$

$x = z^2 - y^2$ → Project onto the zy-plane

$x^2 + y^2 + z^2 = 8$

$x^2 - y^2 + z^2 = 4$ → you choose the plane of projection

$y = 0 \cdot x + z - 2$

$y = 0 \cdot x + 3z + 5$ → project onto xz-plane