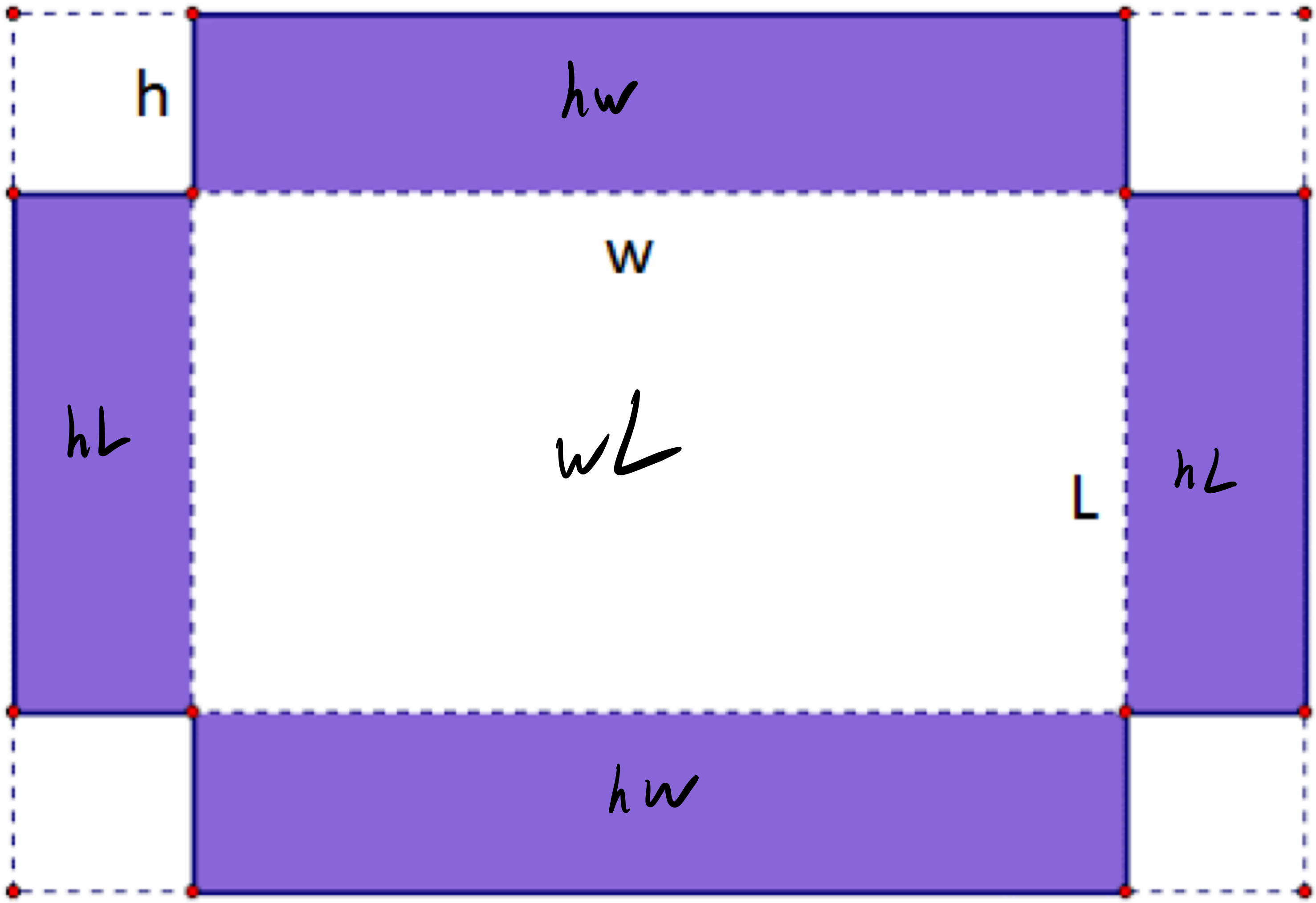
Problem 1: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.



$$\text{area} = f(h,w,L) = wL + 2hw + 2hL$$

$$4\text{m}^3 = \text{Volume} = hwL$$

$$h = \frac{4}{wL} \rightarrow$$

$$f(w,L) = f\left(\frac{4}{wL}, w, L\right) = wL + 2\frac{4}{wL}w + 2\frac{4}{wL}L$$

$$= wL + \frac{8}{L} + \frac{8}{w}.$$

We now want to optimize $f(w,L)$ with no constraints. ($w, L > 0$)

$$f_w(w,L) = L - \frac{8}{w^2}$$

$$f_L(w,L) = w - \frac{8}{L^2}$$

To find C.P.s, we have

$$0 = f_w \quad \Leftrightarrow \quad 0 = L - \frac{8}{w^2} \quad \Leftrightarrow \quad L = \frac{8}{w^2}$$

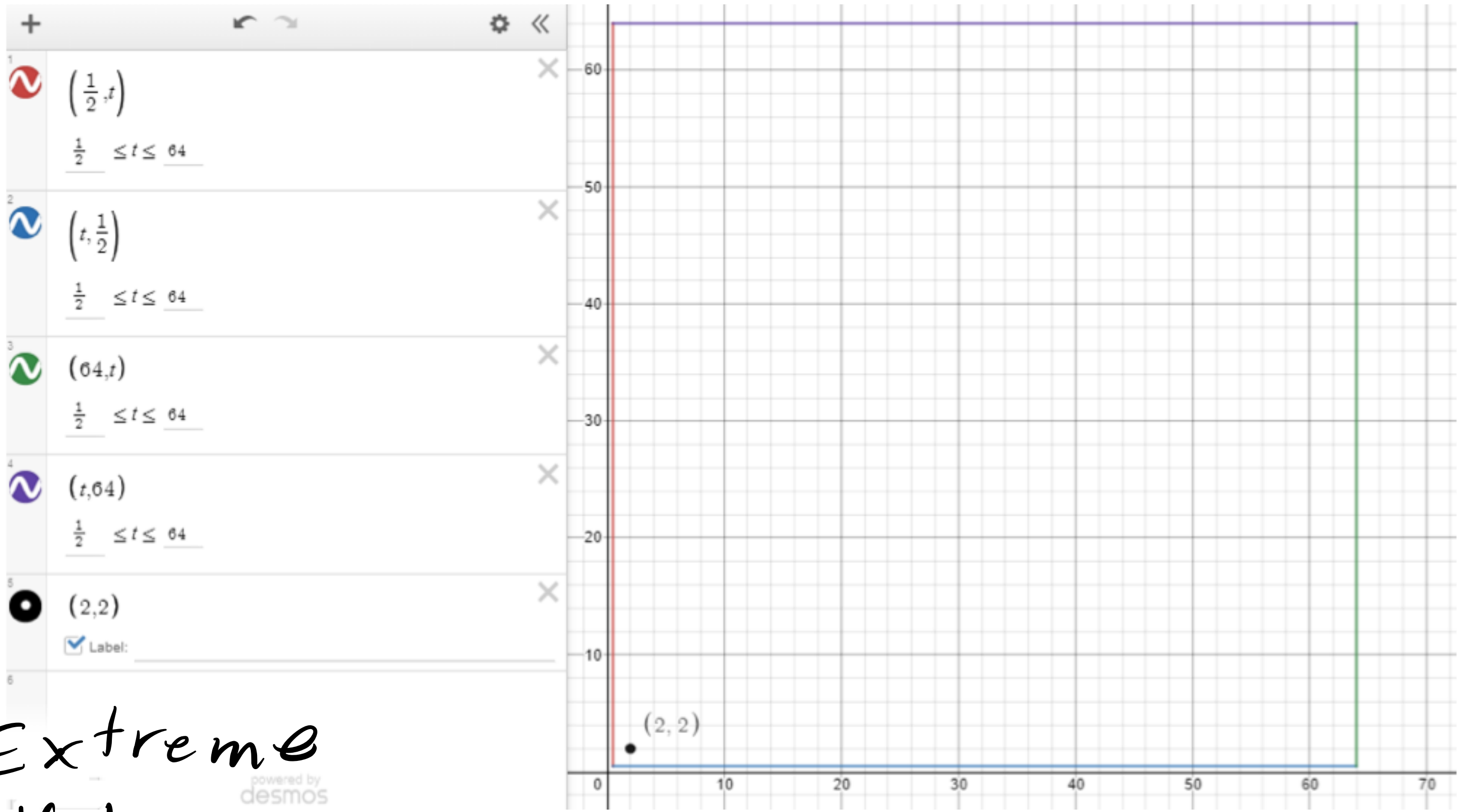
$$0 = f_L \qquad\qquad 0 = w - \frac{8}{L^2} \qquad\qquad w = \frac{8}{L^2}$$

$$\rightarrow \quad w = \frac{8}{\left(\frac{8}{w^2}\right)^2} = \frac{8}{\frac{64}{w^4}}$$

$$= \frac{w^4}{64} \cdot 8 = \frac{w^4}{8}$$

$\rightarrow \quad 8w = w^4, \; w \neq 0 \rightarrow \; 8 = w^3 \rightarrow w = 2.$

$L = \frac{8}{w^2} = 2 \rightarrow (2,2)$ is the only C.P. Note that $f(2,2) = 12$



Extreme Value Theorem: If f is continuous function on a closed and bounded region R, the f attains its absolute max and min over R. Furthermore, these values occur at C.P.s or on the boundary ∂R of R.

Problem 3: Find the absolute minimum and absolute maximum values of the function

Hard Solution

(1) $$f(x, y) = x^2 + 4y^2 + 1$$

over the region

(2) $$R = \{(x, y) : x^2 + 4y^2 \leq 1\}.$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

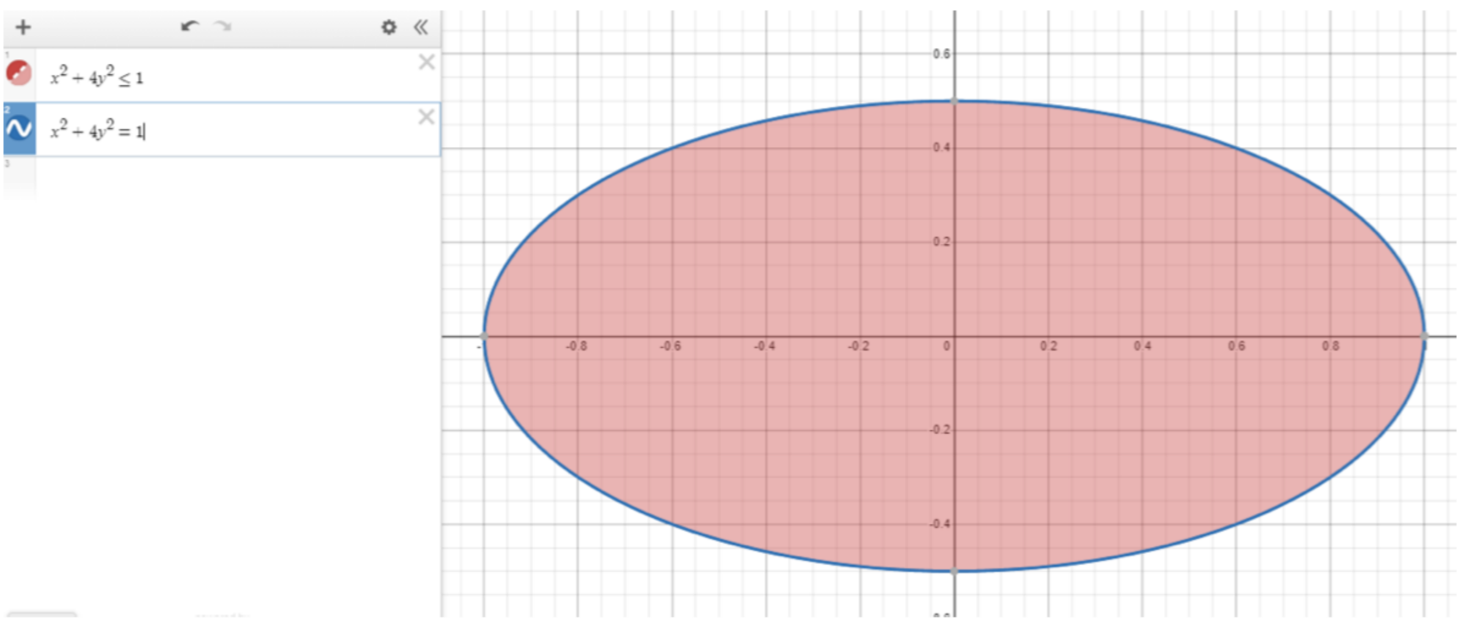


FIGURE 2. The interior of the R is shaded in red and the boundary of R is blue.

Since f is continuous and the region R is closed and bounded we may apply the Extreme Value Thm.

$f(x,y) = x^2 + 4y^2 + 1$

$f_x(x,y) = 2x, \quad f_y(x,y) = 8y$

To find C.P.s we have

$$\begin{matrix} 0 = f_x \\ 0 = f_y \end{matrix} \iff \begin{matrix} 0 = 2x \\ 0 = 8y \end{matrix} \iff \begin{matrix} 0 = x \\ 0 = y \end{matrix}$$

$\rightarrow (0,0)$ is the only C.P.

The boundary of R is

$$\partial R = \{(x,y) \mid x^2 + 4y^2 = 1\}.$$

We use the method of Lagrange multipliers to optimize f on ∂R.

$g(x,y) = x^2 + 4y^2 - 1$

(The constraint is $g(x,y)=0$)

$\vec{\nabla} g(x,y) = \langle 2x, 8y \rangle$

$\vec{\nabla} f(x,y) = \langle 2x, 8y \rangle$, so

$$g(x,y)=0 \qquad \longleftrightarrow$$
$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$

$$x^2+4y^2-1=0$$
$$2x=\lambda 2x$$
$$8y=\lambda 8y$$

if $\lambda=1$ then

$$x^2+4y^2-1=0 \qquad x^2+4y^2-1=0.$$
$$2x=2x \checkmark$$
$$8y=8y \checkmark$$

$\Rightarrow$ The entire boundary consists of C.P.s!

$f(x,y) = x^2 + 4y^2 + 1$

$g(x,y) = x^2 + 4y^2 = 1 \Rightarrow x^2 + 4y^2 = 1$

$f = 1 + 1 = 2$ on ∂R.

$f(0,0) = 1$ so the absolute max of f on R is 2 and the absolute min is 1.

- $f(x,y) = \dfrac{y}{x-5}$ is continuous on

$R_1 = \{(x,y) \mid x^2 + y^2 \leq 1\}$ but not on $R_2 = \{(x,y) \mid x^2 + y^2 \leq 25\}$.

EVT ✓

EVT ✗

$R_3 = \{(x,y) \mid x^2+y^2<25\}$

$\downarrow$

bounded but not closed

~~EVT~~

f is not continuous on R2

Problem 3: Find the absolute minimum and absolute maximum values of the function

Fast Solution

$$f(x, y) = x^2 + 4y^2 + 1 \tag{1}$$

over the region

$$R = \{(x, y) : x^2 + 4y^2 \leq 1\}. \tag{2}$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

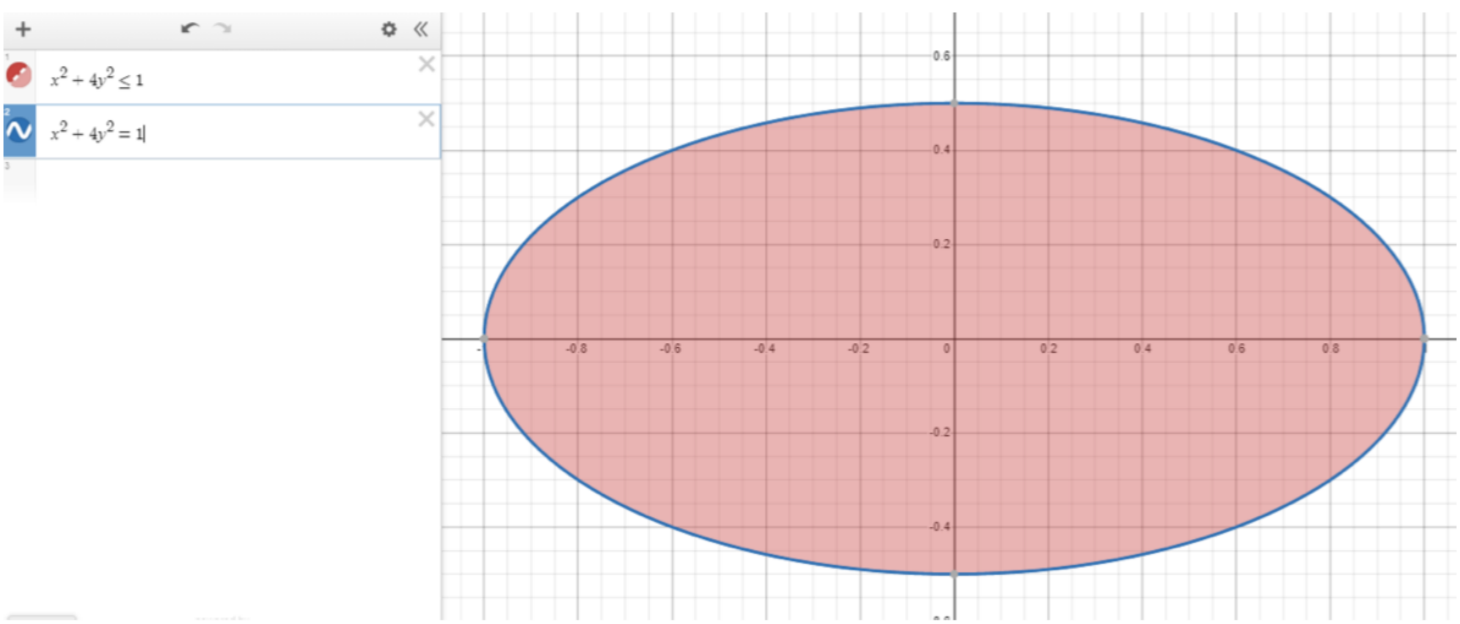


FIGURE 2. The interior of the R is shaded in red and the boundary of R is blue.

Theorem: If f is a continuous function on a closed and bounded region R, then f attains its absolute min and max over R. Furthermore, these values will occur at a critical point of R or on the boundary ∂R of R.

$f(x,y) = x^2 + 4y^2 + 1$

$f_x(x,y) = 2x, \quad f_y(x,y) = 8y$

$$\begin{matrix} 0 = f_x \\ 0 = f_y \end{matrix} \iff \begin{matrix} 0 = 2x \\ 0 = 8y \end{matrix} \iff \begin{matrix} 0 = x \\ 0 = y \end{matrix}$$

$\rightarrow$ $(0,0)$ is the only C.P.

Now we handle the boundary $\partial R = \{(x,y) \mid x^2 + 4y^2 = 1\}$, on which $f(x,y) = 1 + 1 = 2$.

Since $f(0,0) = 1$, we see that 2 is the absolute max which occurs on all of ∂R. and 1 is the absolute min. which occurs at $(0,0)$.

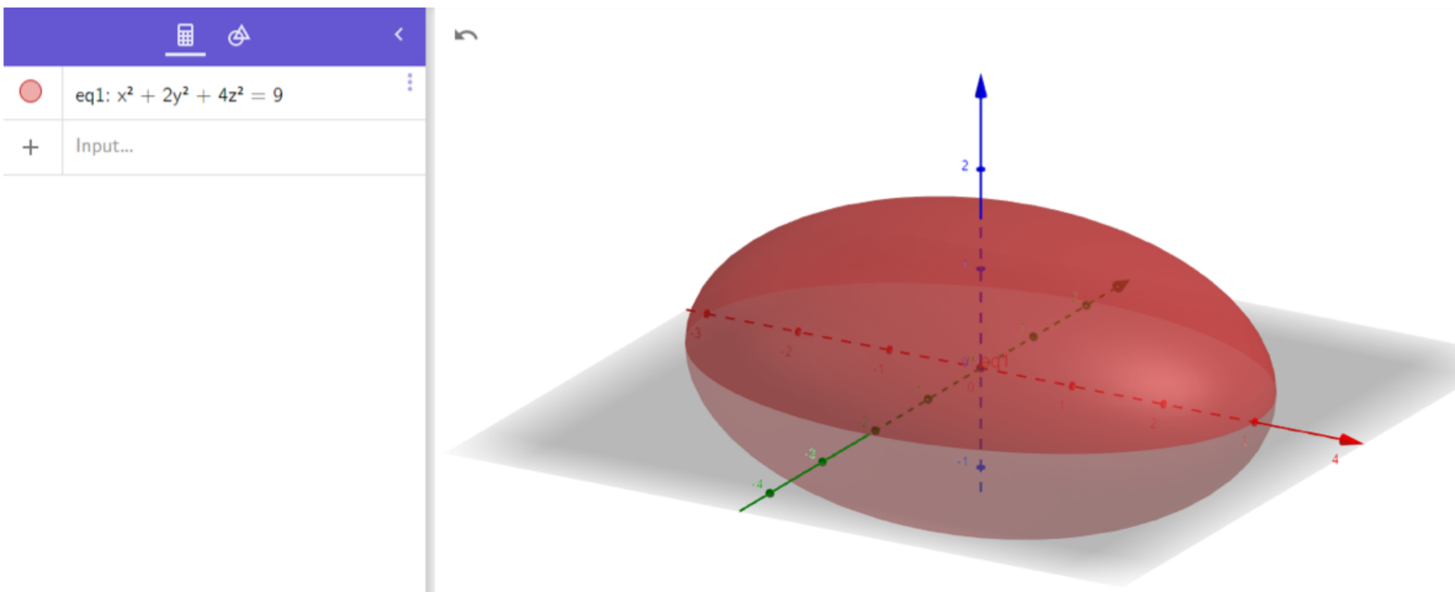
Problem 4: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

$$f(x, y, z) = xyz \tag{3}$$

subject to the constraint

$x^2+2y^2+4z^2-9=0$

$$x^2 + 2y^2 + 4z^2 = 9. \tag{4}$$



$g(x,y,z) = x^2+2y^2+4z^2-9$ is the constraint function.

$\vec{\nabla} g(x,y,z) = \langle 2x, 4y, 8z \rangle$

$\vec{\nabla} f(x,y) = \langle yz, xz, xy \rangle$

The method of Lagrange multipliers gives us the system

$$g(x,y,z)=0 \qquad \longleftrightarrow$$
$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

$$\boxed{\begin{array}{c} x^2+2y^2+4z^2-9=0 \\ yz = 2x\lambda \\ xz = 4y\lambda \\ xy = 8z\lambda \end{array}}$$

we want to divide equations to cancel λ, but we can't.

We use cross multiplication and factoring.

$$2x\lambda\, xz = yz\, 4y\lambda \rightarrow$$
$$2\lambda x^2 z = 4y^2 z\lambda \rightarrow$$
$$2\lambda x^2 z - 4\lambda y^2 z = 0 \rightarrow$$
$$2\lambda z(x^2-2y^2) = 0 \rightarrow$$ By the zero-product property

$\lambda=0$ or $z=0$ or $x^2-2y^2=0$.

Case 1: $\lambda = 0$.

$\lambda = 0$

$x^2 + 2y^2 + 4z^2 - 9 = 0$

$yz = 0 \to y$ or $z = 0$

$xz = 0 \to x$ or $z = 0$

$xy = 0 \to x$ or $y = 0$

$\to$ 2 of x, y, z are 0. $\to$ so

$y, z = 0 \to x^2 - 9 = 0 \to (\pm 3, 0, 0)$, or

$x, z = 0 \to 2y^2 - 9 = 0 \to (0, \pm \frac{3}{\sqrt{2}}, 0)$, or

$x, y = 0 \to 4z^2 - 9 = 0 \to (0, 0, \pm \frac{3}{2})$.

We found 6 C.P.s in Case 1.

Case 2: $z = 0$, $\lambda \neq 0$.

$z = 0$

$x^2 + 2y^2 - 9 = 0$

$0 = 2\lambda x \to x = 0$ (b/c $\lambda \neq 0$)

$0 = 4\lambda y \to y = 0$

$xy = 0$

$\to (x, y, z) = (0, 0, 0)$ does not satisfy

So no C.P.s are obtained.

Case 3: $x^2 - 2y^2 = 0$, $\lambda \neq 0$, $z \neq 0$.

$x^2 = 2y^2$

$x = \pm\sqrt{2}\,y \Rightarrow x = +\sqrt{2}y$ and $x = -\sqrt{2}y$ are our 2 subcases.

If $x = \sqrt{2}\,y$ then

$$(\sqrt{2}y)^2 + 2y^2 + 4z^2 - 9 = 0$$

$x = \sqrt{2}y$

$$yz = 2\lambda\sqrt{2}y$$

$$\sqrt{2}y\,z = 4\lambda y \Rightarrow yz = 2\sqrt{2}\lambda y$$

$$\sqrt{2}yy = 8\lambda z$$

$\rightarrow$

$$2y^2 + 2y^2 + 4z^2 = 9$$

$$yz - 2\sqrt{2}\lambda y = 0 \Rightarrow y(z - 2\sqrt{2}\lambda) = 0$$

$$y^2 = 4\sqrt{2}\lambda z \quad \downarrow \text{ By Z.P.P.}$$

$$y = 0 \quad \text{or} \quad z = 2\sqrt{2}\lambda$$

Subsubcase I: $y = 0$.

$4z^2 = 9$ $\qquad (\lambda, z \neq 0)$

$0 = 4\sqrt{2}\lambda z \Rightarrow$ no solns.

Subsubcase 2: $z = 2\sqrt{2}\lambda$

$y^2 = 4\sqrt{2}\lambda \, 2\sqrt{2}\lambda = 16\lambda^2$

$4y^2 + 4(2\sqrt{2}\lambda)^2 = 9$

$4y^2 + 64\lambda^2 = 9$

$y^2 + 16\lambda^2 = \frac{9}{4}$

$y^2 + y^2 = \frac{9}{4}$ & $\lambda = \pm\frac{\sqrt{4}}{4}$?

$\rightarrow 2y^2 = \frac{9}{4} \Rightarrow y = \frac{\pm 3}{2\sqrt{2}}$

$\rightarrow (x,y,z) = \left(\pm\frac{3}{2}, \pm\frac{3}{2\sqrt{2}}\right)$

Problem 5: What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.

For (x,y,z) on our plane, the distance to the origin $(0,0,0)$ is

$$f(x,y,z) = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$
$$= \sqrt{(8-y-4z)^2+y^2+z^2}$$
$$= \sqrt{64-16y-64z+8yz+y^2+16z^2+y^2+z^2}$$
$$= \sqrt{64-16y-64z+8yz+2y^2+17z^2} := f(y,z)$$

So we want to optimize $f(y,z)$ with no constraints. Since x^2 is a strictly increasing function on $[0,\infty)$, $f(y,z)$ and $h(y,z) := f(y,z)^2$ have their minimums and maximums in the same locations, so we choose to optimize $h(y,z)$ instead.

$$h(y,z) = 64 - 16y - 64z + 8yz + 2y^2 + 17z^2$$

We start by finding the C.P.s.

$$h_y(y,z) = -16 + 8z + 4y$$

$$h_z(y,z) = -64 + 8y + 34z$$

$$\begin{array}{l} 0 = h_y \\ 0 = h_z \end{array} \longleftrightarrow \begin{array}{l} 0 = -16 + 8z + 4y \\ 0 = -64 + 8y + 34z \end{array}$$

$$0 = 0 - 2 \cdot 0 = \underline{-64 + 8y + 34z} - 2(\underline{-16 + 8z + 4y})$$

$$= -64 + \cancel{8y} + 34z + 32 - 16z - \cancel{8y}$$

$$= -32 + 18z$$

$$\to 18z = 32$$

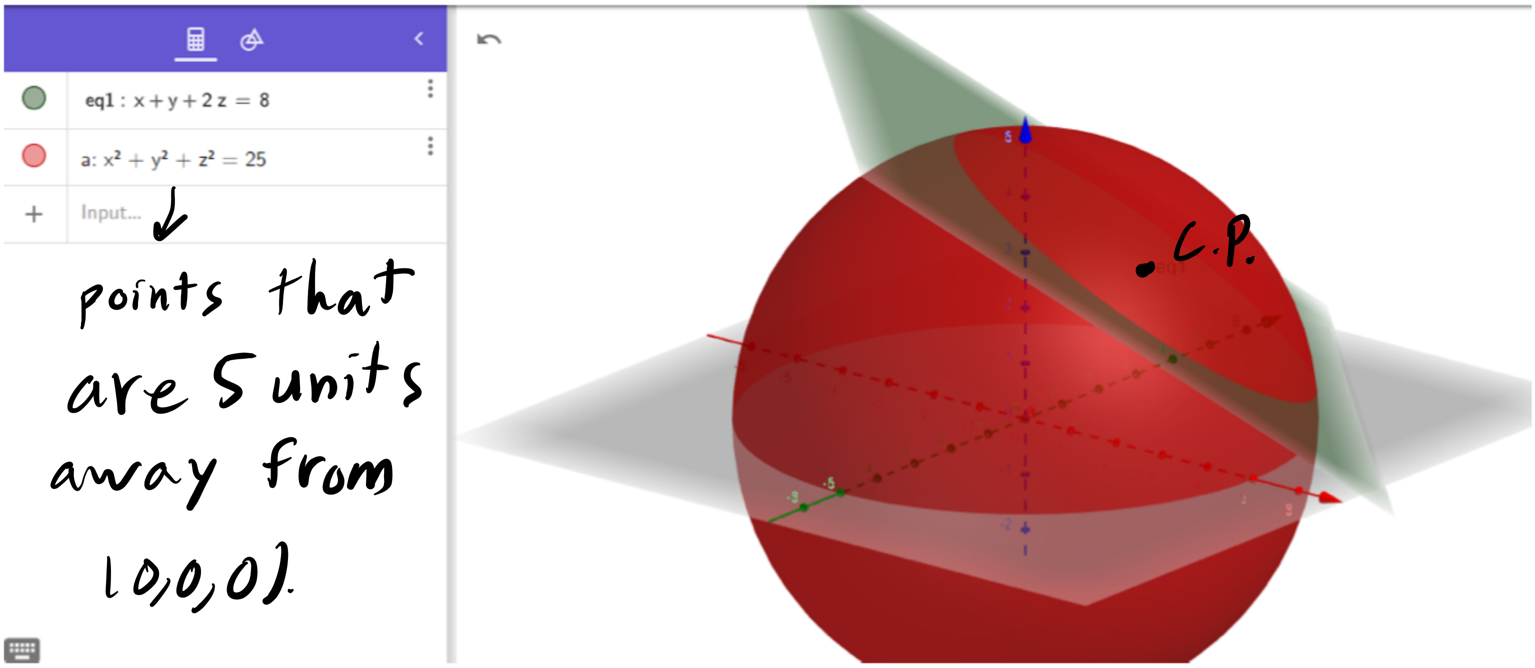
$$\to z = \frac{32}{18} = \frac{16}{9}.$$

$$0 = -16 + 8 \cdot \frac{16}{9} + 4y$$

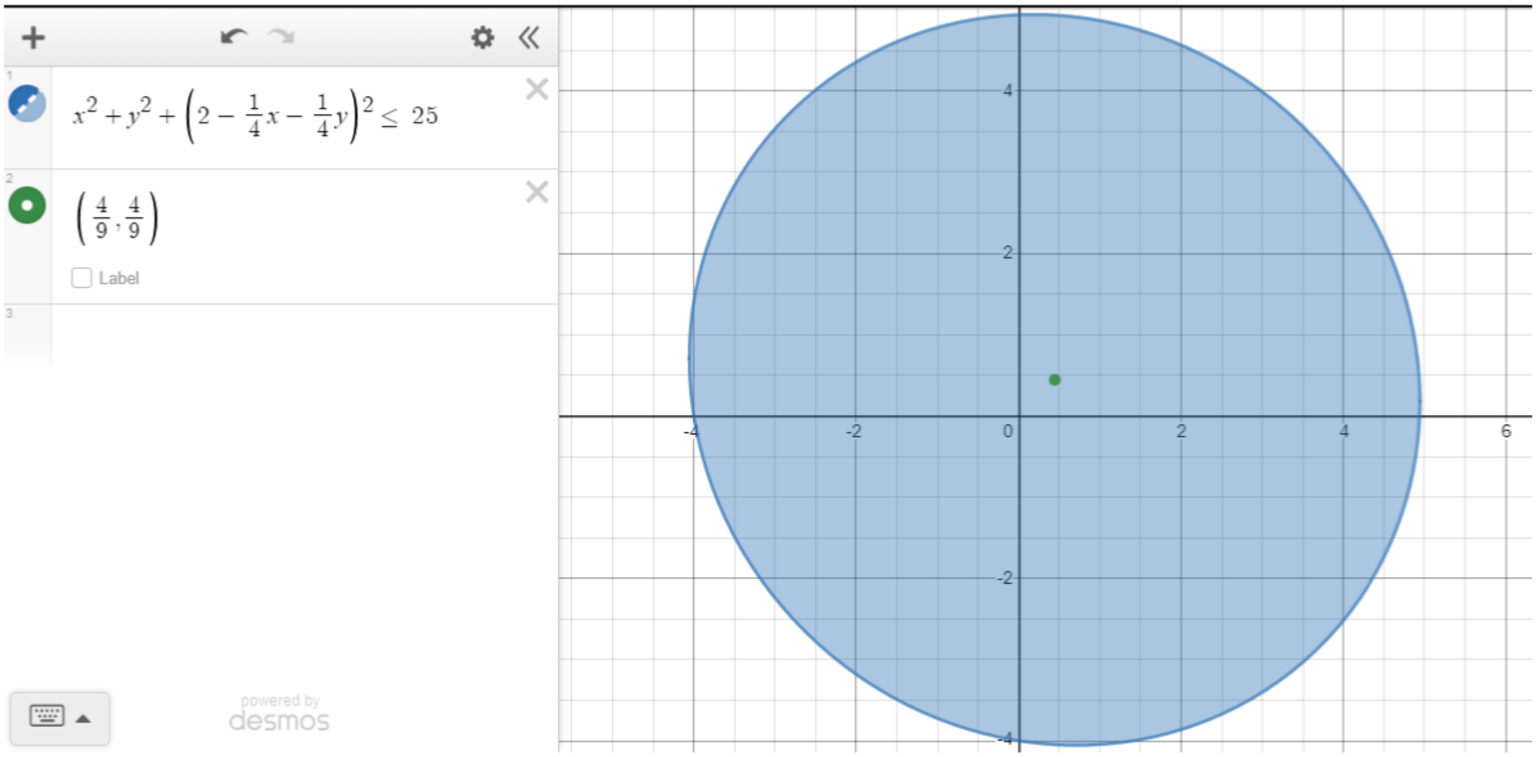
$$\to 0 = -4 + 2 \cdot \frac{16}{9} + y \to y = 4 - \frac{32}{9} = \frac{4}{9}$$

$\Rightarrow$ $\left(\frac{4}{9}, \frac{16}{9}\right)$ is the only C.P.

$$h\left(\frac{4}{9}, \frac{16}{9}\right) = \_\_\_\_ < 5$$



$h(y,z) \le 25$



Theorem: a continuous function f on a closed and bounded region R will attain its absolute max and min either at a C.P. in R or on the boundary ∂R of R.

Problem 7: Given the production function $P = f(K, L) = K^a L^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1-a)B/q$. (Recall that $K \geq 0$ and $L \geq 0$ in order for the model to make sense in the real world and in order for the production function f to be well defined.)

- p is price per unit of capitol
- K is the amount of capitol
- q is price per unit of Labor
- L is the amount of Labor
- B is the budget.

(Constraint equation is $g(K,L)=0$)

$g(K,L) = pK + qL - B$ is our constraint function.

$f(K,L) = K^a L^{1-a}$

$\vec{\nabla} g(K,L) = \langle p, q \rangle$

$\vec{\nabla} f(K,L) = \langle aK^{a-1}L^{1-a}, (1-a)K^a L^{-a} \rangle$

The method of Lagrange multipliers gives

$g(K,L) = 0$

$\vec{\nabla} f(K,L) = \lambda \vec{\nabla} g(K,L)$

$pk + qL - B = 0$

$ak^{a-1}L^{1-a} = \lambda p$

$(1-a)k^{a}L^{-a} = \lambda q$

$\lambda q a k^{a-1}L^{1-a} = \lambda p(1-a)k^{a}L^{-a}$

$\lambda q a k^{a-1}L^{1-a} - \lambda p(1-a)k^{a}L^{-a} = 0$

multiply by $L^{a}k^{1-a}$

$\lambda q a L - \lambda p(1-a)k = 0$

$\lambda(qaL - p(1-a)k) = 0 \Rightarrow$ Z.P.P.

$\lambda = 0$ or $qaL - p(1-a)k = 0$

Case 1: If $\lambda = 0$

$\lambda = 0$

$ak^{a-1}L^{1-a} = 0 \Rightarrow$ no

$(1-a)k^{a}L^{-a} = 0$ solns.

Case 2: $qaL - p(1-a)k = 0$

$$\rightarrow \quad L = \frac{p\|-a\|k}{qa} \rightarrow \text{done.}$$

$$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(x,y) = g(s,t)$$

$$s = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

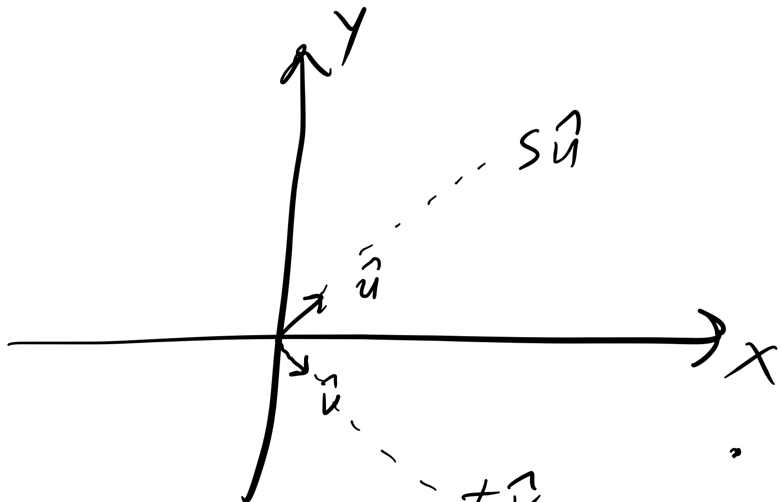
$$z = xy$$

$$t = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

$$(a,b) = (1,1)$$

$$\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$\hat{v} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$



(x,y)

$= s\hat{u} + t\hat{v}$

$=$ "(s,t)"