Problem 14.3.5.57: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) If the speed of an object is constant, then its velocity components are constant.

(b) The functions $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ and $\vec{R}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ generate the same set of points for $t \geq 0$. ~~(Bonus: What about for $t \geq \pi^2$?)~~

(c) A velocity vector (vector valued function) of variable magnitude cannot have constant direction.

(d) If the acceleration of an object is $\vec{a}(t) = \vec{0}$, for all $t \geq 0$, then the velocity of the object is constant.

(e) If you double the initial speed of a projectile, its range also double (assume no forces other than gravity).

(f) If you double the initial speed of a projectile, its time of flight also doubles (assume no forces other than gravity).

(g) A trajectory with $\vec{v}(t) = \vec{a}(t) \neq \vec{0}$, for all t, is possible.

---

a) False. Consider

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, \ 0 \leq t \leq 2\pi$$

$$\rightarrow \vec{v}(t) = \vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle \neq \text{constant}$$

$$\rightarrow \text{Speed} = |\vec{v}(t)| = \sqrt{(-\sin(t))^2 + \cos^2(t)} = 1$$

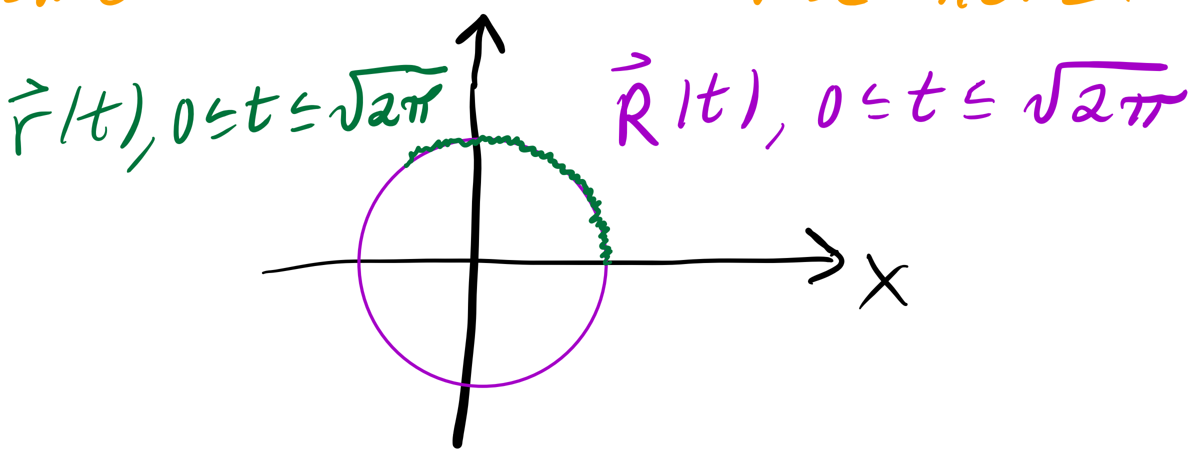
---

b) True. Note that for any $t \geq 0$, $\sqrt{t} \geq 0$ and

$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle = \langle \cos(\sqrt{t}^{\,2}), \sin(\sqrt{t}^{\,2}) \rangle = \vec{R}(\sqrt{t})$. Similarly, $\vec{R}(t) = \vec{r}(t^2)$, and $t^2 \geq 0$ when $t \geq 0$.

- Every point of $\vec{r}(t)$ is also a point of $\vec{R}(t)$.
- Every point of $\vec{R}(t)$ is also a point of $\vec{r}(t)$.

- Note, if $0 \le t \le \sqrt{2\pi}$, then $\vec{R}(t)$ traces out the entire unit circle while $\vec{r}(t)$ traces out part of the unit circle, so the answer would be false here.



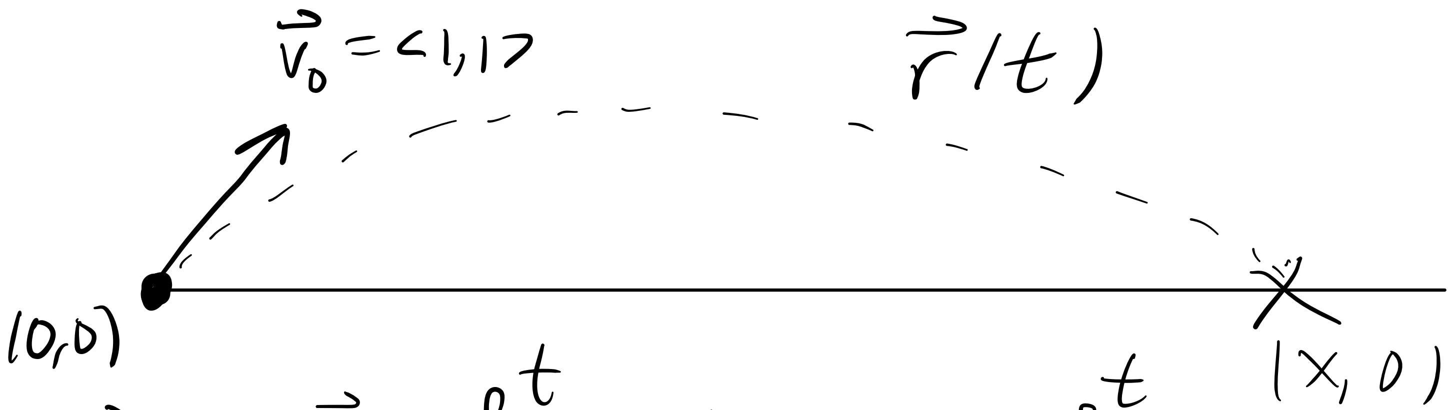
c) False. Consider $\vec{v}(t) = \langle t, t \rangle = t\langle 1, 1 \rangle, t \ge 1$.

- The direction of $\vec{v}(t)$ is the same as that of $\langle 1, 1 \rangle$, so it is constant.
- $|\vec{v}(t)| = \sqrt{t^2 + t^2} = \sqrt{2}\, t$ is variable.

d) $\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)\,du = \vec{v}_0 + \int_0^t \langle 0, 0 \rangle du = \vec{v}_0 + \langle 0, 0 \rangle = \vec{v}_0$

$\rightarrow$ True.

e) False. Consider a projectile launched with initial velocity $\vec{v}_0 = \langle 1,1 \rangle$ and let us approximate gravity by $\vec{a} = \langle 0,-10 \rangle$ (meters/seconds^2).



$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)\,du = \langle 1,1 \rangle + \int_0^t \langle 0,-10 \rangle\,du$$

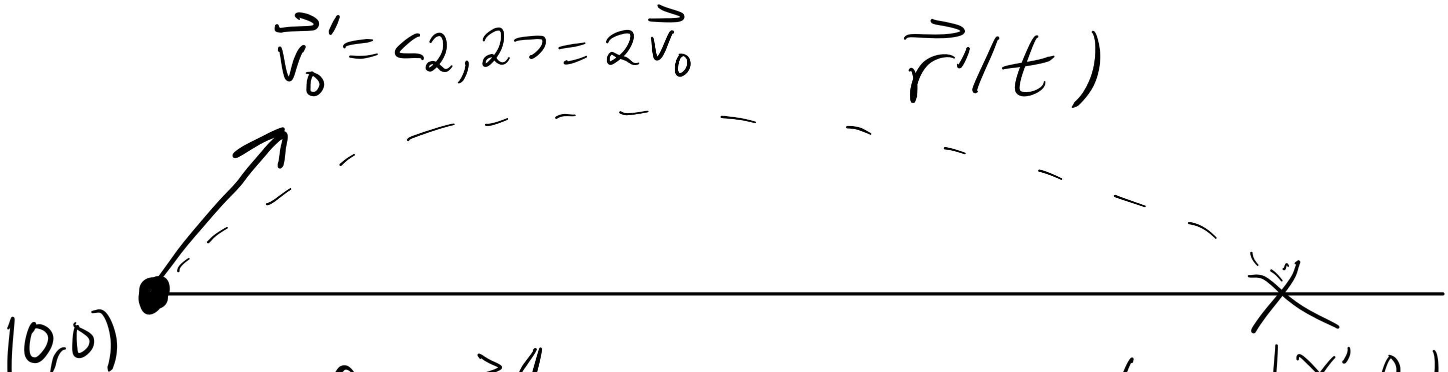
$$= \langle 1,1 \rangle + \langle 0,-10t \rangle = \langle 1, 1-10t \rangle.$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)\,du = \langle 0,0 \rangle + \int_0 \langle 1, 1-10u \rangle\,du$$

$$= \langle t, t-5t^2 \rangle$$

$$\vec{r}(t) = \langle x,0 \rangle \Rightarrow t-5t^2 = 0 \Rightarrow t = \cancel{0}, \tfrac{1}{5}$$

$$x = t = \tfrac{1}{5}.$$



Now if $\vec{v}_0{}' = \langle 2,2\rangle$, then

$$\vec{v}\,'(t) = \vec{v}_0{}' + \int_0^t \vec{a}(u)\,du = \langle 2,2\rangle + \int_0^t \langle 0,-10\rangle\,du$$

$$= \langle 2,2\rangle + \langle 0,-10t\rangle = \langle 2, 2-10t\rangle$$

$$\vec{r}\,'(t) = \vec{r}_0{}' + \int_0^t \vec{v}\,'(u)\,du = \langle 0,0\rangle + \int_0^t \langle 2, 2-10u\rangle\,du$$

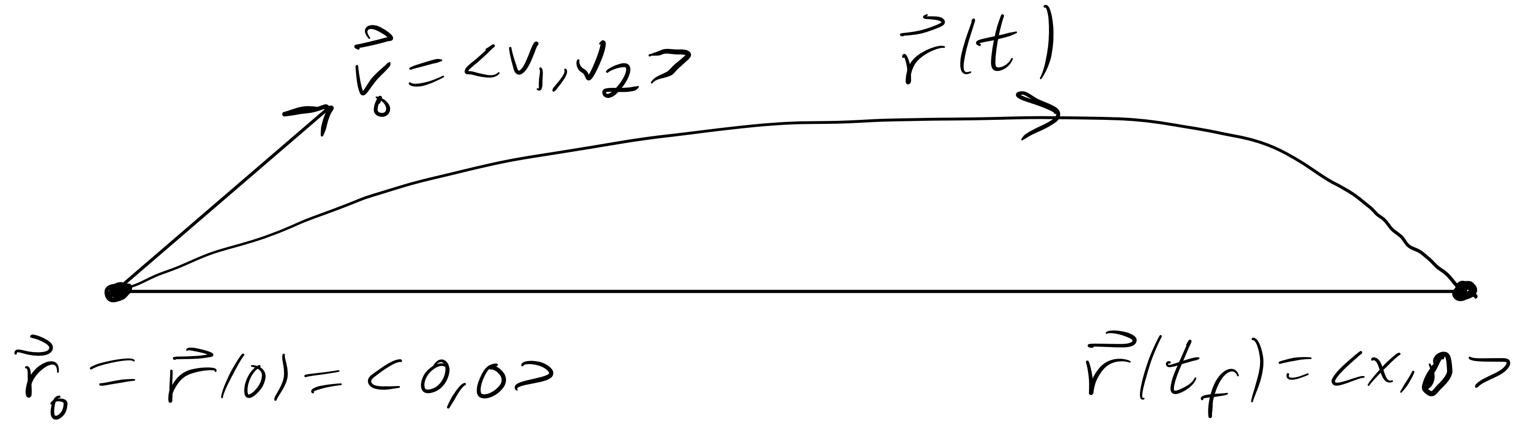
$$= \langle 2t, 2t-5t^2\rangle$$

$$\vec{r}\,'(t) = \langle x', 0\rangle \Rightarrow 2t-5t^2 = 0 \Rightarrow t = \cancel{0}, \frac{2}{5}$$

$$x' = 2t = \frac{4}{5} \neq 2x$$

Bonus: In this example, doubling the initial velocity quadrupled the range. Can you show that this always happens?

f) True. Let $\vec{v}_0 = \langle v_1, v_2 \rangle$.



as in part g,

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)\,du = \langle v_1, v_2 \rangle + \int_0^t \langle 0, -10 \rangle\,du$$

$$= \langle v_1, v_2 \rangle + \langle 0, -10t \rangle = \langle v_1, v_2 - 10t \rangle$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)\,du = \langle 0, 0 \rangle + \int_0^t \langle v_1, v_2 - 10u \rangle\,du$$

$$= \langle v_1 t, v_2 t - 5t^2 \rangle$$

$\vec{r}(t_f) = \langle x, 0 \rangle \rightarrow \; v_2 t_f - 5t_f^2 = 0 \rightarrow t_f = \cancel{0}, \dfrac{v_2}{5}$.

For $\vec{v}_0' = 2\vec{v}_0 = \langle v_1', v_2' \rangle = \langle 2v_1, 2v_2 \rangle$, we have

$$t_f' = \frac{v_2'}{5} = \frac{2v_2}{5} = 2t_f.$$

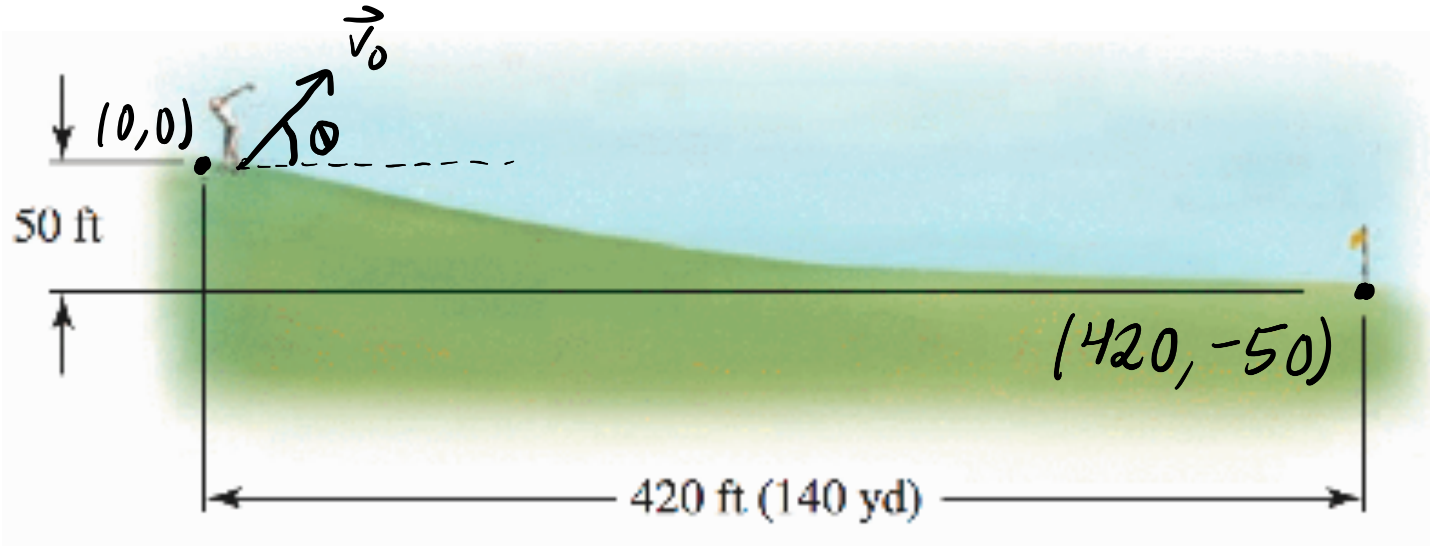
g) True. If $\vec{r}(t) = \langle e^t, e^t \rangle$, then $\vec{v}(t) = \vec{r}'(t) = \langle e^t, e^t \rangle$, and $\vec{a}(t) = \vec{v}'(t) = \langle e^t, e^t \rangle$, so

$\vec{a}(t) = \vec{v}(t)$ for all t, and

$\vec{a}(t) \neq \vec{0}$ for all t since

e^t is never 0.

Problem 14.3.5.67: A golfer stands 420ft (140yd) horizontally from the hole and 50ft above the hole (see figure). Assuming the ball is hit with an initial speed of120ft/s, at what angle(s) should it be hit to land in the hole? Assume the path of the ball lies in a plane.



$|\vec{v}_0| = 120\ ft/s$, $\vec{r}_0 = \langle 0,0 \rangle$

$\vec{v}_0 = \langle 120\cos(\theta), 120\sin(\theta) \rangle$

$\vec{a}(t) = \langle 0, -g \rangle \approx \langle 0, -32 \rangle.$

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)\,du = \vec{v}_0 + \int_0^t \langle 0, -32 \rangle\,du$$

$$= \vec{v}_0 + \langle 0, -32u|_0^t \rangle = \vec{v}_0 + \langle 0, -32t \rangle$$

$$= \langle 120\cos(\theta), 120\sin(\theta) - 32t \rangle$$

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)\,du = \langle 0,0 \rangle + \int_0^t \vec{v}(u)\,du = \int_0^t \vec{v}(u)\,du$$

$$=\int_0^t \langle \underbrace{120\cos(\theta)}_{\text{constant}}, \underbrace{120\sin(\theta)}_{\text{constant}}-32u \rangle du$$

$$=\langle 120\cos(\theta)u\Big|_{u=0}^{t}, 120\sin(\theta)u-16u^2\Big|_{u=0}^{t} \rangle$$

$$=\langle 120\cos(\theta)t, 120\sin(\theta)t-16t^2 \rangle$$

Now, we want to solve for t, θ s.t.

$$\langle 420, -50 \rangle = \vec{r}(t)$$

$$=\langle 120\cos(\theta)t, 120\sin(\theta)t-16t^2 \rangle \rightarrow$$

$$120\cos(\theta)t = 420$$

$$120\sin(\theta)t-16t^2=-50$$

$$\rightarrow \cos(\theta) = \frac{420}{120t} = \frac{7}{2t} \rightarrow$$

$$\sin(\theta) = \pm\sqrt{1-\cos^2(\theta)} = \sqrt{1-\left(\frac{7}{2t}\right)^2}$$

$$=\sqrt{\frac{4t^2-49}{4t^2}} = \frac{\sqrt{4t^2-49}}{2t}$$

$$\rightarrow -50 = 120\frac{\sqrt{4t^2-49}}{2t}t - 16t^2$$

$$= 60\sqrt{4t^2-49} - 16t^2 \rightarrow$$

$$16t^2 - 50 = 60\sqrt{4t^2-49} \rightarrow$$

$$(16t^2-50)^2 = 60^2(4t^2-49) \rightarrow$$

$$256t^4 - 1600t^2 + 2500 = 14400t^2 - 3600\cdot 49$$

$$\rightarrow \underbrace{256}_{a}t^4 \underbrace{-160000}_{b}t^2 + \underbrace{2500+3600\cdot 49}_{c} = 0$$

$$a(t^2)^2 + bt^2 + c = 0$$

$$t^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow$$

$$t = \pm\sqrt{\frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

2 solns

$$\rightarrow \boxed{\theta = \cos^{-1}\left(\frac{7}{2t}\right)}$$

Problem 14.4.3.43: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) If an object moves on a trajectory with constant speed S over a time interval $a \leq t \leq b$, then the length of the trajectory is $S(b-a)$.

(b) The curves defined by

$$\vec{r}(t) = \langle f(t), g(t)\rangle \text{ and } \vec{R}(t) = \langle g(t), f(t)\rangle \tag{1}$$

have the same length over the interval $[a, b]$.

(c) The curve $\vec{r}(t) = \langle f(t), g(t)\rangle$, for $0 < a \leq t \leq b$, and the curve $\vec{R}(t) = \langle f(t^2), g(t^2)\rangle$, for $\sqrt{a} \leq t \leq \sqrt{b}$, have the same length.

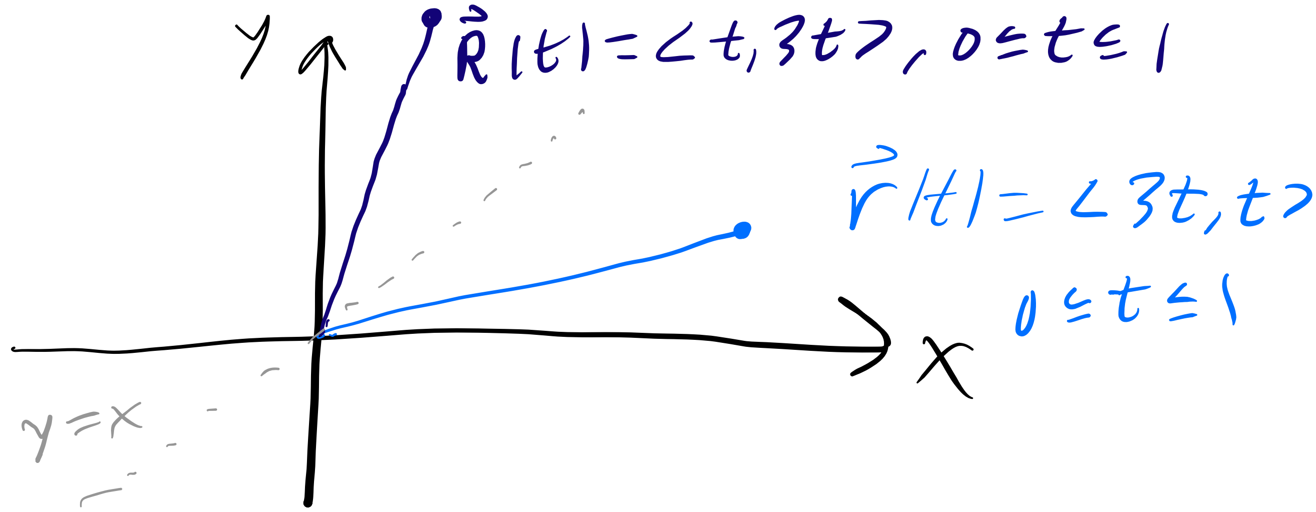
(d) The curve $\vec{r}(t) = \langle t, t^2, 3t^2\rangle$, for $1 \leq t \leq 4$, is parameterized by arclength.

a) True. $S = \text{speed} = |\vec{v}(t)|$.

Length of trajectory $= \int_a^b \underbrace{|\vec{v}(t)|}_{=|\vec{r}\,'(t)|} dt$

$= \int_a^b S\,dt = St\Big|_{t=a}^{b} = S(b-a)$.

b) True. Reason 1: $\vec{R}(t)$ is the reflection of $\vec{r}(t)$ over the line $y=x$, and reflection preserves length.



Reason 2: Direct calculation.

$$\text{arclength}(\vec{r}) = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$= \int_a^b |\langle f'(t), g'(t)\rangle\, dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt, \text{ and}$$

$$\text{arclength}(\vec{R}) = \int_a^b |\vec{R}\,'(t)|\,dt$$

$$= \int_a^b |\langle g'(t), f'(t)\rangle|\, dt$$

$$= \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}\, dt =$$

c) True. Reason 1: Direct Calculation
(left to the students.
Hint: Use u-substitution)

Reason 2: $\vec{r}$ and $\vec{R}$ represent the same curve.

We see that $\vec{r}(t) = \langle f(t), g(t) \rangle$
$= \langle f(\sqrt{t}^2), g(\sqrt{t}^2) \rangle = \vec{R}(\sqrt{t})$
for $a \le t \le b$, since $\sqrt{a} \le \sqrt{t} \le \sqrt{b}$.

→ every point on $\vec{r}$ is a point on $\vec{R}$, so $\vec{r}$ is inside of $\vec{R}$.

Similarly, $\vec{R}(t) = \vec{r}(t^2)$ for all $\sqrt{a} \le t \le \sqrt{b}$ since $a \le t^2 \le b$.

→ every point on $\vec{R}$ is a point on $\vec{r}$, so $\vec{R}$ is inside of $\vec{r}$.

→ $\vec{r}$ and $\vec{R}$ are the same curve.

d) False. $\vec{r}(t) = \langle t, t^2, 3t^2 \rangle$, $1 \le t \le 4$,

$\vec{r}\,'(t) = \langle 1, 2t, 6t \rangle$,

$|\vec{r}\,'(t)| = \sqrt{1^2 + (2t)^2 + (6t)^2} = \sqrt{1 + 40t^2}$

$\neq 1$ if $t \neq 0$.

Problem 14.4.3.46 Consider the curve $\mathcal{C}$ that is described by the parameterization $\vec{r}(t) = \langle t^m, t^m, t^{\frac{3}{2}m}\rangle$ where $0 \leq a \leq t \leq b$ and $m \neq 0$.

(a) Find the arclength function $s(t)$. Note that your answer may include a, b, and m in it.

(b) Find the parameterization by arclength for $\mathcal{C}$ when $a = \frac{29}{9}, b = 4$, and $m = 2$.

---

a) $\vec{r}'(t) = \langle mt^{m-1}, mt^{m-1}, \frac{3}{2}mt^{\frac{3}{2}m-1}\rangle,$

$$|\vec{r}'(t)| = \sqrt{(mt^{m-1})^2 + (mt^{m-1})^2 + \left(\frac{3}{2}mt^{\frac{3}{2}m-1}\right)^2}$$

$$= \sqrt{m^2t^{2m-2} + m^2t^{2m-2} + \frac{9}{4}m^2t^{3m-2}}$$

$$= \sqrt{m^2t^{2m-2}\left(1+1+\frac{9}{4}t^m\right)}$$

$$= mt^{m-1}\sqrt{2+\frac{9}{4}t^m}$$

$$s = s(t) = \int_a^t |\vec{r}'(u)|\,du = \int_a^t mu^{m-1}\sqrt{2+\frac{9}{4}u^m}\,du$$

$v = u^m$

$dv = mu^{m-1}du$

$$\int_{u=a}^{t} \sqrt{2+\frac{9}{4}v}\,dv = \frac{8}{27}\left(2+\frac{9}{4}v\right)^{\frac{3}{2}}\Big|_{u=a}^{t}$$

$$=\frac{8}{27}\left(2+\frac{9}{4}u^m\right)^{\frac{3}{2}}\Big|_{u=a}^{t}$$

$$=\frac{8}{27}\left(2+\frac{9}{4}t^m\right)^{\frac{3}{2}}-\frac{8}{27}\left(2+\frac{9}{4}a^m\right)^{\frac{3}{2}},$$

$a \le t \le b$

b) $m=2,\ a=\frac{28}{9},\ b=4 \rightarrow$

$$s=s(t)=\frac{8}{27}\left(2+\frac{9}{4}t^2\right)^{\frac{3}{2}}-\frac{8}{27}\left(2+\frac{9}{4}\cdot\frac{28}{9}\right)^{\frac{3}{2}}$$

$$=\frac{8}{27}\left(2+\frac{9}{4}t^2\right)^{\frac{3}{2}}-\frac{8}{27}(\underbrace{2+7}_{9})^{\frac{3}{2}}$$

$$=\frac{8}{27}\left(2+\frac{9}{4}t^2\right)^{\frac{3}{2}}-8 \rightarrow$$

$$s+8=\frac{8}{27}\left(2+\frac{9}{4}t^2\right)^{\frac{3}{2}} \Rightarrow$$

$$\frac{27}{8}s+27=\left(2+\frac{9}{4}t^2\right)^{\frac{3}{2}} \rightarrow$$

$$\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}=2+\frac{9}{4}t^2 \rightarrow$$

$$\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-2=\frac{9}{4}t^2 \rightarrow$$

$$\frac{4}{9}\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-\frac{8}{9}=t^2 \rightarrow$$

$$t=\sqrt{\frac{4}{9}\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-\frac{8}{9}}$$

$\rightarrow$ The parameterization by arclength is

$$\vec{r}(t)=\vec{r}(t(s))=\langle t(s)^2, t(s)^2, t(s)^3 \rangle$$

$$=\left\langle \frac{4}{9}\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-\frac{8}{9},\ \frac{4}{9}\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-\frac{8}{9},\ \left(\frac{4}{9}\left(\frac{27}{8}s+27\right)^{\frac{2}{3}}-\frac{8}{9}\right)^{\frac{3}{2}} \right\rangle$$