Problem 15.2.5.82: Verify that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = 1. \tag{1}$$

$$\sin(x)+\sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y}$$

$$= \lim_{(x,y)\to(0,0)} \frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{x+y}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{\frac{x+y}{2}}$$

$z = \frac{x+y}{2}$

$$= \underbrace{\left(\lim_{\substack{(x,y)\to(0,0)\\ \Downarrow\\ z\to 0}} \frac{\sin\left(\frac{x+y}{2}\right)}{\frac{x+y}{2}}\right)}_{1} \underbrace{\left(\lim_{(x,y)\to(0,0)} \cos\left(\frac{x-y}{2}\right)\right)}_{\cos\left(\frac{0-0}{2}\right)=1}$$

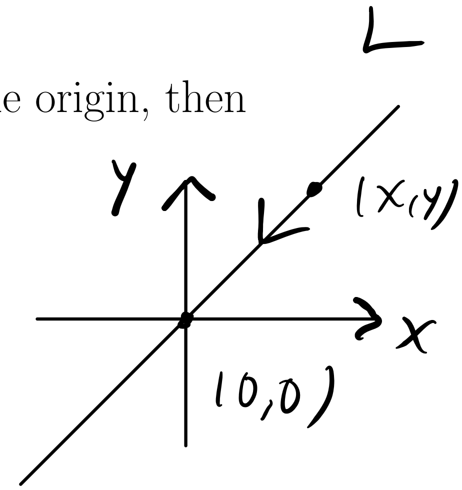
$$= 1 \cdot 1 = 1 \checkmark$$

Homemade Problem: Consider the function

$$f(x,y) = \frac{xy^2}{x^2+y^4}. \tag{2}$$

(a) Show that if L is a line that passes through the origin, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = 0. \tag{3}$$



(b) Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \tag{4}$$

does not exist.

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a) Case 1: The line L has the equation $y = mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \lim_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4}$$

$$= \lim_{x \to 0} \frac{m^2 x}{1+m^4x^2} = \frac{m^2 \cdot 0}{1+m^4 \cdot 0^2} = 0.$$

Case 2: The equation of L is $x = 0$.

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in L}} f(x,y) = \lim_{y \to 0} f(0,y)$$

$$= \lim_{y\to 0} \frac{0 \cdot y^2}{0+y^4}$$

$$= \lim_{y \to 0} \frac{0}{y^4} = \lim_{y\to 0} 0 = 0.$$

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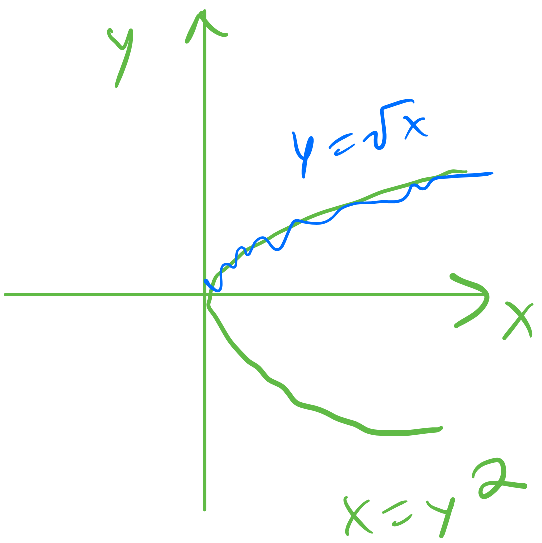
b) We use the path P $x = y^2$.

We see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P}} f(x,y) = \lim_{y\to 0} f(\underbrace{y^2}_{x},y)$$

$$= \lim_{y\to 0} \frac{(\overbrace{y^2}^{x})(y)^2}{(\underbrace{y^2}_{x})^2+(y)^4}$$



$$= \lim_{y\to 0} \frac{y^4}{y^4+y^4}$$

$$= \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since the path $y=x$ and the path $x=y^2$ yield the values of 0 and $\frac{1}{2}$ respectively for $\lim\limits_{\substack{(x,y)\to(0,0)\\(x,y)\in P}} f(x,y)$, we see that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE

Problem 15.3.97: Consider the function $f(x,y) = \sqrt{|xy|}$.

(a) Is f continuous at $(0,0)$?

$\boxed{\text{(b) Is } f \text{ differentiable at } (0,0)?}$ (d)

b (c) Show that $f_x(0,0)$ and $f_y(0,0)$ exist by calculating their values.

c (d) Determine whether f_x and f_y are continuous at $(0,0)$.

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a) Yes, f is continuous everywhere.

The composition of continuous functions is continuous when it is defined.

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b) $$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0}$$

$$= \lim_{x\to 0} \frac{\sqrt{|x\cdot 0|} - \sqrt{|0\cdot 0|}}{x-0}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = \lim_{x\to 0} 0 = 0.$$

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0}$$

$$= \lim_{y\to 0} \frac{\sqrt{|0\cdot y|} - \sqrt{|0\cdot 0|}}{y-0}$$

$$= \lim_{y\to 0} \frac{0-0}{y} = \lim_{y\to 0} 0 = 0.$$

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c) We will restrict our attention to $x,y>0$ (Q I), so $f(x,y) = \sqrt{|xy|} = \sqrt{xy}$.

$$f_x(x,y) = \frac{\partial}{\partial x}\left(\sqrt{x\cancel{y}_2}\right) \overset{\substack{\text{chain}\\\text{rule}\\\downarrow}}{=} \frac{1}{2\sqrt{x\cancel{y}_2}} \cdot \frac{\partial}{\partial x}(x\cancel{y}_2)$$

pretend $y=2$

$$= \frac{1}{2\sqrt{x\cancel{y}_2}} \cdot \cancel{y}_2 = \frac{1}{2\sqrt{xy}} \cdot y = \frac{1}{2}\sqrt{\frac{y}{x}}$$

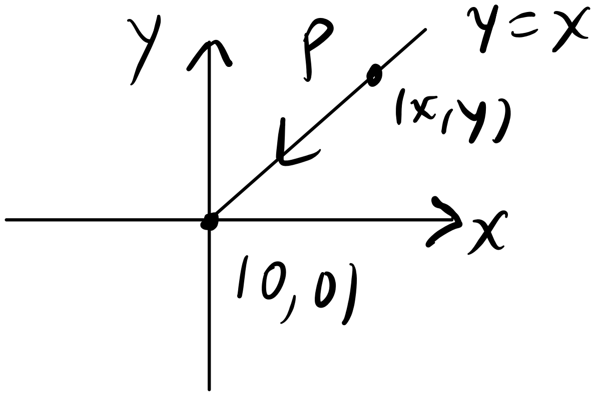
$$f_y(x,y) = \frac{\partial}{\partial y}\left(\sqrt{xy}\right) \overset{\text{chain rule}}{=} \frac{1}{2\sqrt{xy}} \cdot \frac{\partial}{\partial y}(xy)$$

$$= \frac{1}{2\sqrt{xy}} \cdot x = \frac{1}{2\sqrt{xy}} \cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}$$

We will show that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) \neq f_x(0,0) = 0$$

to see that f_x is not continuous at $(0,0)$. It suffices to show that there is a single path for which the limit is not 0.



$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P}} f_x(x,y) = \lim_{x\to 0^+} f_x(x,x)$$

$$= \lim_{x\to 0^+} \frac{1}{2}\sqrt{\frac{x}{x}} = \lim_{x\to 0^+} \frac{1}{2}$$

$$= \frac{1}{2} \neq 0 = f_x(0,0),$$

So f_x is not continuous at $(0,0)$.
a similar calculation shows that f_y is not continuous at $(0,0)$.

Theorem: If f is differentiable at $(0,0)$, and $\hat{u}$ is a direction, then the directional derivative

$$D_{\vec{u}} f(0,0) := \lim_{t\to 0} \frac{f((0,0)+t\hat{u}) - f(0,0)}{t}$$

can also be calculated by

$$D_{\vec{u}} f(0,0) = \vec{\nabla} f(0,0) \cdot \hat{u}$$

$$= \langle f_x(0,0), f_y(0,0) \rangle \cdot \hat{u}$$

Observe that

$f_x(0,0) = D_{\langle 1,0 \rangle} f(0,0)$ and

$f_y(0,0) = D_{\langle 0,1 \rangle} f(0,0)$.

Consider $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. Note

$$\vec{\nabla} f(0,0) \cdot \hat{u}$$

$$= \langle f_x(0,0), f_y(0,0) \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle.$$

$$= \langle 0, 0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0.$$

However,

$$D_{\hat{u}} f(0,0) = \lim_{t \to 0} \frac{f((0,0)+t\hat{u}) - f(0,0)}{t}$$

$$= \lim_{t \to 0} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - 0}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{\left|\frac{t}{\sqrt{2}} \cdot \frac{t}{\sqrt{2}}\right|}}{t}$$

$$= \lim_{t \to 0} \frac{\sqrt{\left|\frac{t^2}{2}\right|}}{t} = \lim_{t \to 0} \frac{\sqrt{\frac{t^2}{2}}}{t}$$

$$= \lim_{t \to 0} \frac{\frac{t}{\sqrt{2}}}{t} = \lim_{t \to 0} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0,$$

so f is not differentiable.