**Review Problem 1.92:** What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.

**Solution:** Note that for any $(x, y, z)$ on the plane $x + y + 4z = 8$ we have

$$z = 2 - \frac{1}{4}x - \frac{1}{4}y, \tag{1}$$

from which we see that

$$\mathrm{d}((x, y, z), (0, 0, 0)) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \tag{2}$$

$$= \sqrt{x^2 + y^2 + (2 - \frac{1}{4}x - \frac{1}{4}y)^2} = \sqrt{4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2}. \tag{3}$$

We recall that if $f(x, y)$ is any nonnegative function, then $f(x, y)$ and $f^2(x, y)$ have their (local and global) minimums and maximums occur at the same values of $(x, y)$. It follows that we want to optimize the function

$$f(x, y) = 4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2. \tag{4}$$

Since any global minimum of $f(x, y)$ is also a local minimum, we see that the global minimum of $f$ (if it exists) is at a critical point. We now begin finding the critical points of $f$. We see that

$$\begin{array}{l} 0 = f_x(x, y) = \frac{17}{8}x + \frac{1}{8}y - 1 \\ 0 = f_y(x, y) = \frac{17}{8}y + \frac{1}{8}x - 1 \end{array} \rightarrow 0 = (\frac{17}{8}x + \frac{1}{8}y - 1) - (\frac{17}{8}y + \frac{1}{8}x - 1) \tag{5}$$

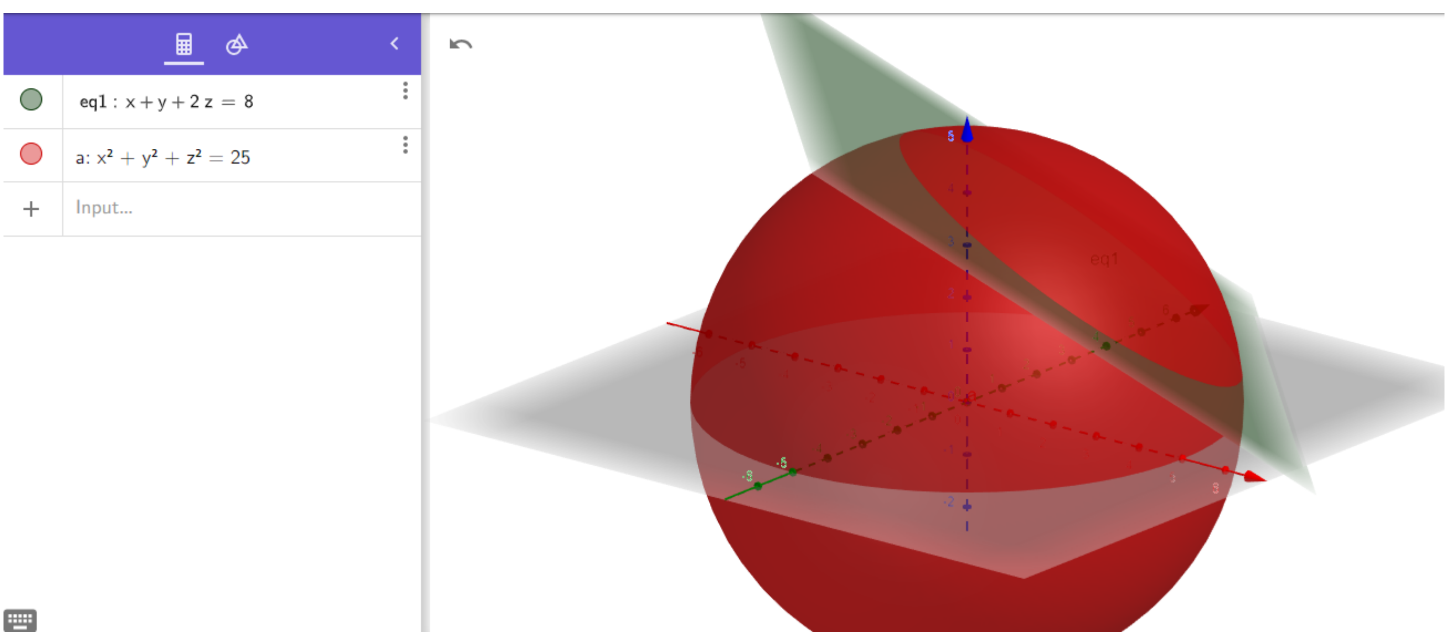
$$= 2x - 2y \rightarrow x = y \rightarrow x = y = \frac{4}{9}. \tag{6}$$

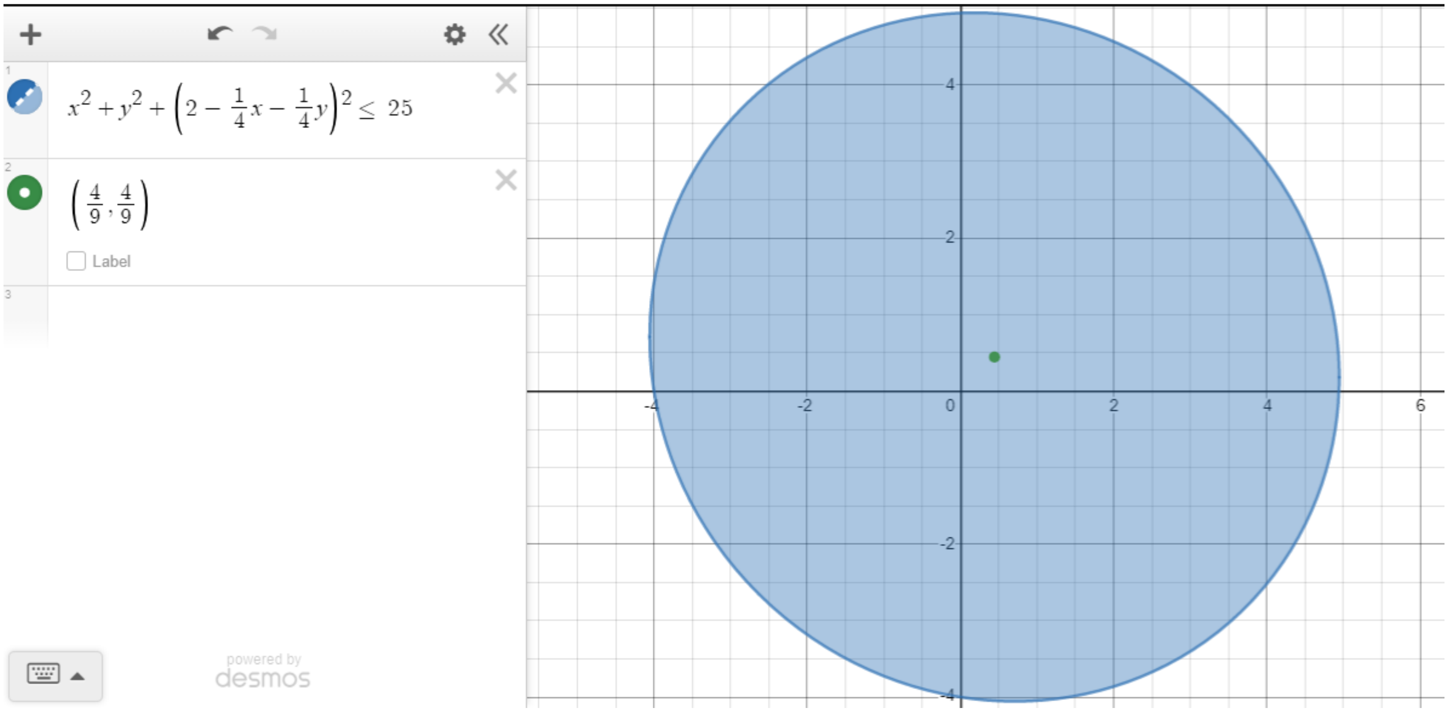
We see that $(\frac{4}{9}, \frac{4}{9})$ is the only critical point. We will now use the second derivative test to verify that $(\frac{4}{9}, \frac{4}{9})$ is a local minimum. We see that

$$\begin{aligned} f_{xx}(x,y) &= \tfrac{17}{8} \\ f_{yy}(x,y) &= \tfrac{17}{8} \rightarrow D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 \\ f_{xy}(x,y) &= \tfrac{1}{8} \end{aligned} \tag{7}$$

$$= \frac{17}{8}\cdot\frac{17}{8} - (\frac{1}{8})^2 = \frac{9}{2} \rightarrow D(\frac{4}{9},\frac{4}{9}) = \frac{9}{2} > 0. \tag{8}$$

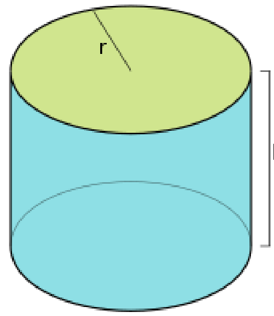
Since we also see that $f_{xx}(\frac{4}{9},\frac{4}{9}) = \frac{17}{8} > 0$, the second derivative test tells us that $(\frac{4}{9},\frac{4}{9})$ is indeed a local minimum of $f(x,y)$. It remains to show that $f(x,y)$ attains its global minimum at $(\frac{4}{9},\frac{4}{9})$. Firstly, we note that $f(\frac{4}{9},\frac{4}{9}) = \frac{4\sqrt{2}}{3}$. Since $\frac{32}{9} < 25$ (I picked 25 randomly, I just needed some larger number), let us consider the region $R$ of $(x,y)$ for which $(x,y,\underbrace{2-\frac{1}{4}x-\frac{1}{4}y}_{z})$ has a distance of at most 5 from the origin. This is the same as $R = \{(x,y) \mid f(x,y) \leq 25\}$.






Since $R$ is a closed and bounded region, and $f(x, y)$ is a continuous function function, we know that $f$ attains an absolute minimum on $R$. The point $(\frac{4}{9}, \frac{4}{9})$ is inside of $R$, so the minimum of $f$ is not attained on the boundary of $R$ (as that is where the distance to the origin is exactly 5). Since the minimum of $f$ on $R$ is attained on the interior, we see that it must be obtained at a critical point of $f(x, y)$, so it is attained at $(\frac{4}{9}, \frac{4}{9})$. For any point $(x, y)$ outside of $R$, we have $f(x, y) > 25$ (by the very definition of $R$), so the global minimum of $f(x, y)$ is $\frac{32}{9}$ and is attained at $(\frac{4}{9}, \frac{4}{9})$. It follows that the point on the plane $x + y + 2z = 8$ that is closest to the origin is $\boxed{(\frac{4}{9}, \frac{4}{9}, \frac{16}{9})}$.

**Review Problem 1.98:** Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of $32\pi$ in$^3$.




**Solution:** We recall that a cylinder of radius $r$ and height $h$ has a volume of $V = \pi r^2 h$ and a surface area (including the 2 circular ends) of $S = 2\pi r^2 + 2\pi rh$. It follows that we want to optimize the function $f(r,h) = 2\pi r^2 + 2\pi rh$ subject to the constraint $0 = g(r,h) = \pi r^2 h - 32\pi$. Since

$$\nabla f(r,h) = \langle 4\pi r + 2\pi h, 2\pi r\rangle \text{ and } \nabla g(r,h) = \langle 2\pi rh, \pi r^2\rangle, \text{ we obtain} \tag{9}$$

$$\begin{array}{rcl} 4\pi r + 2\pi h & = & 2\pi\lambda rh \\ 2\pi r & = & \pi\lambda r^2 \\ \pi r^2 h & = & 32\pi \end{array} \overset{r\neq 0}{\rightarrow} \begin{array}{rcl} 2r + h & = & \lambda rh \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \rightarrow \begin{array}{rcl} 2r + h & = & 2h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \tag{10}$$

$$\rightarrow \begin{array}{rcl} 2r & = & h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \rightarrow \begin{array}{rcl} 2r & = & h \\ 2 & = & \lambda r \\ 2r^3 & = & 32 \end{array} \rightarrow r = \sqrt[3]{16} = 2\sqrt[3]{2} \rightarrow h = 4\sqrt[3]{2}. \tag{11}$$

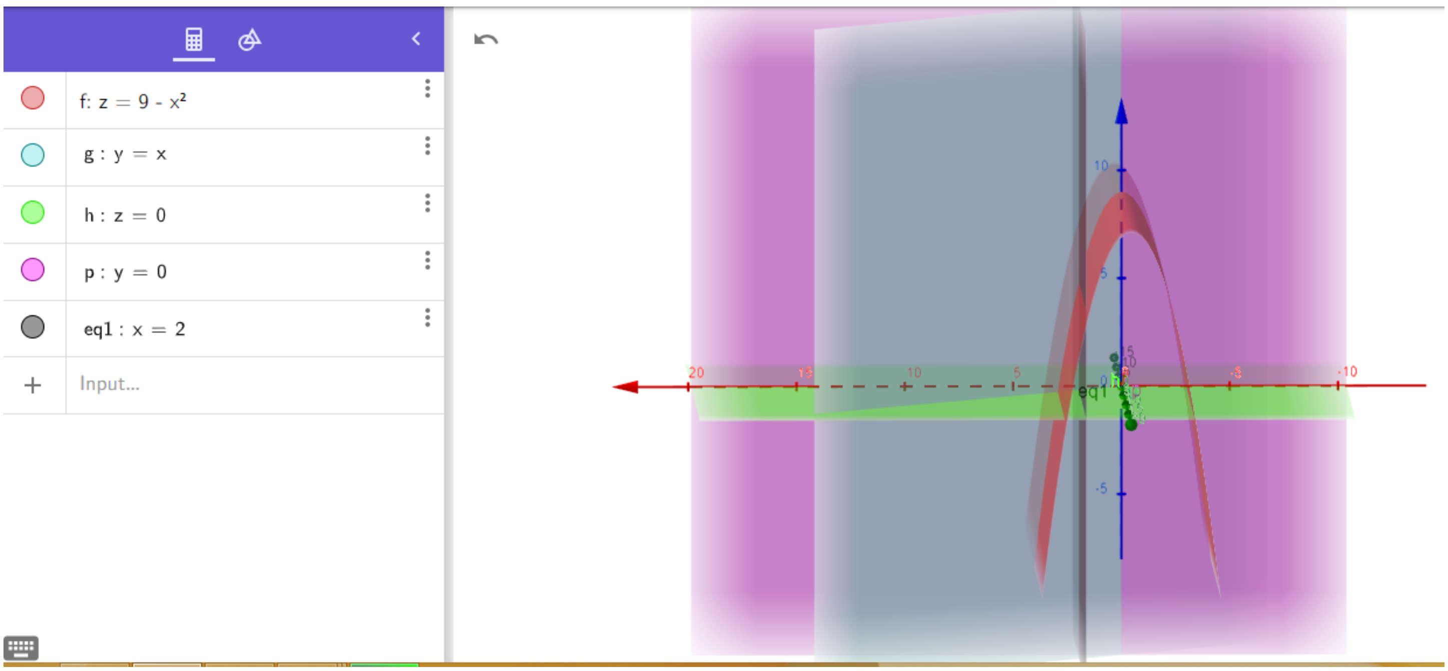
Since the cylinder does not have a maximum surface area when subjected to the constraint $V = 32\pi$, we see that the critical point that we found has to correspond to a local minimum. The extreme/boundary cases occur when either $r \to \infty$ or $h \to \infty$, in which case we also have $S \to \infty$. It follows that $f(r,h)$ attains a minimum value of $24\pi\sqrt[3]{4}$ when $(r,h) = \boxed{(2\sqrt[3]{2}, 4\sqrt[3]{2})}$.

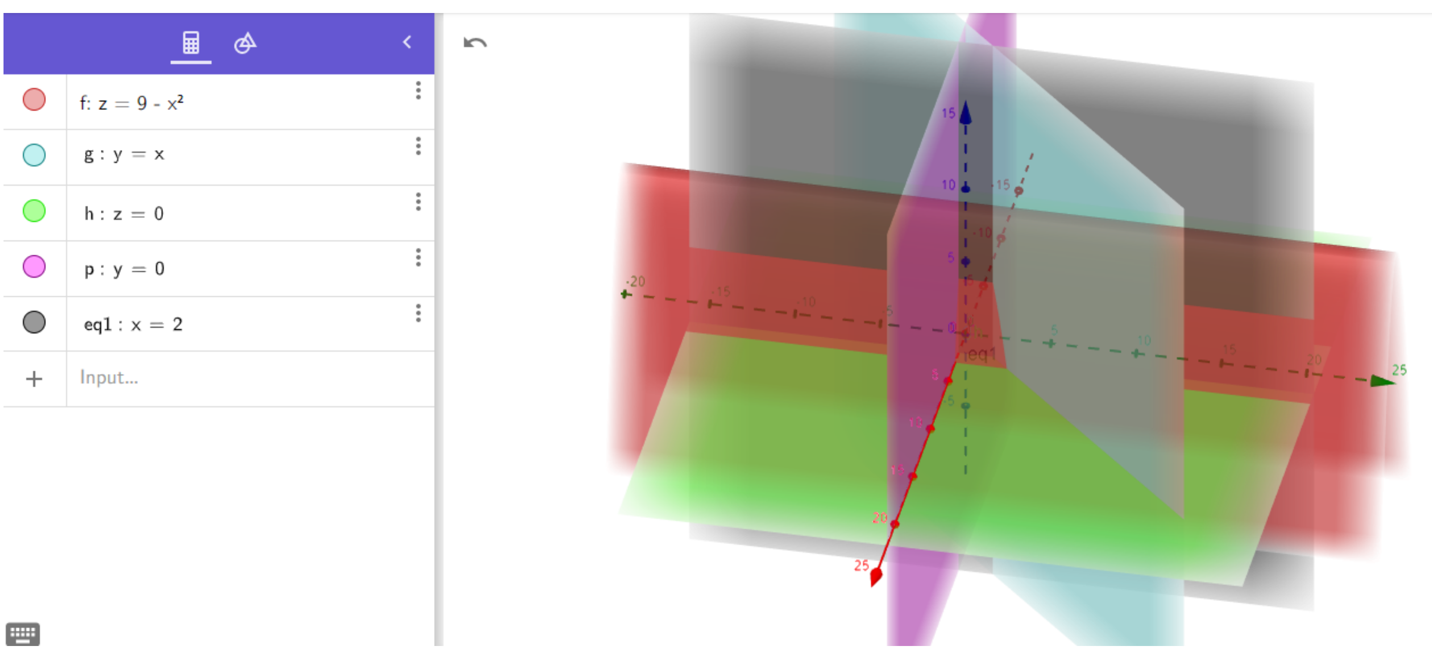
**Review Problem 2.26:** Rewrite the the triple integral

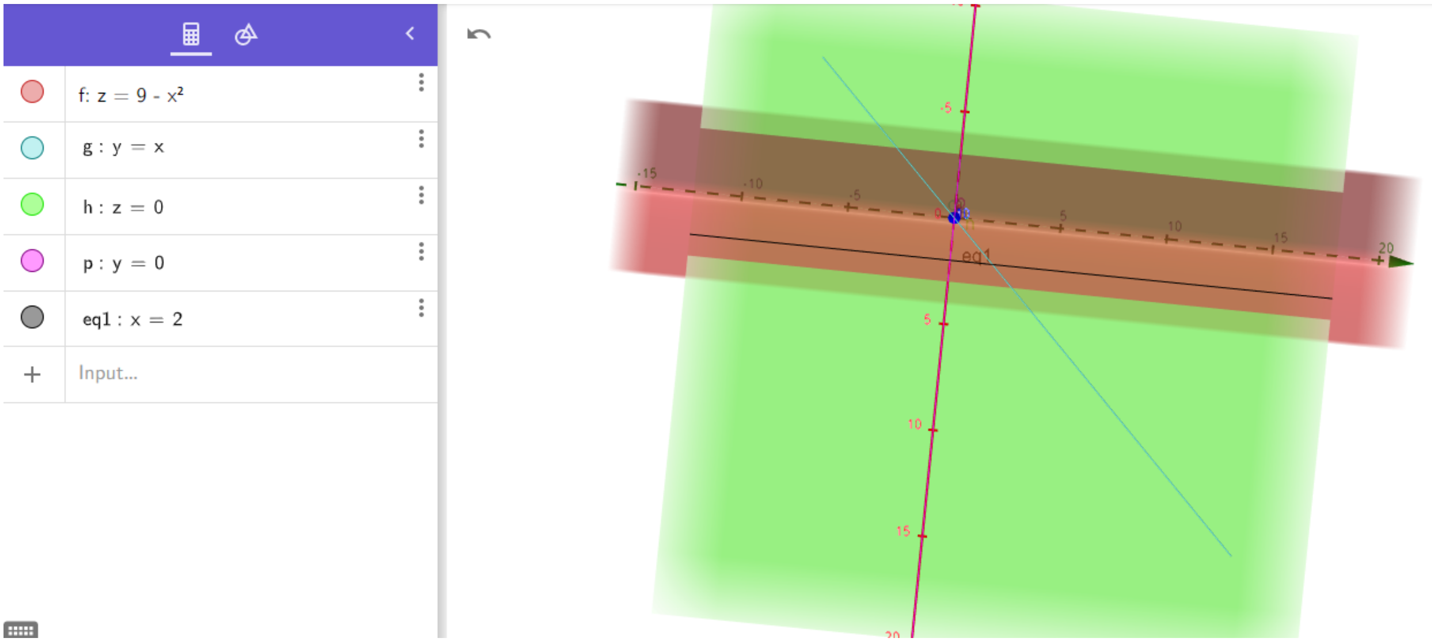
$$\int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)\,dy\,dz\,dx \tag{12}$$

using the order $dzdxdy$.

**First Solution:** We envision the 3-dimensional solid that is described by the bounds of the triple integral in the currect order of $dydzdx$, and then we traverse the solid using the new order of $dzdxdy$.

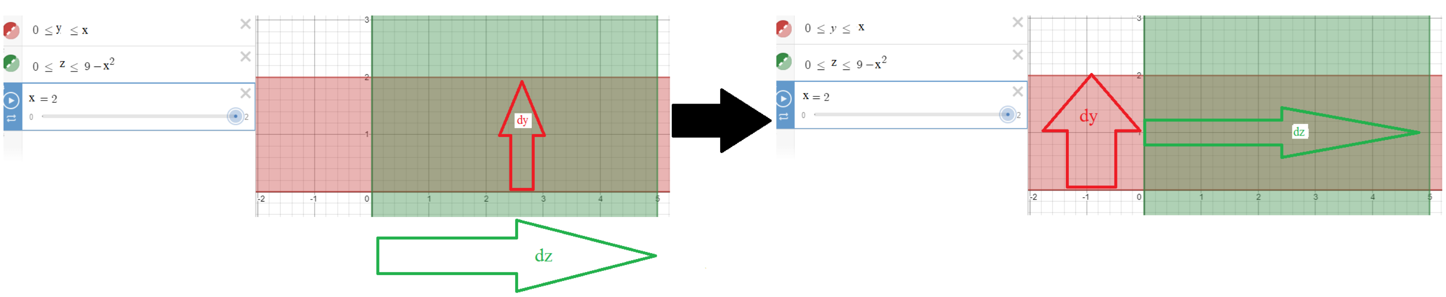


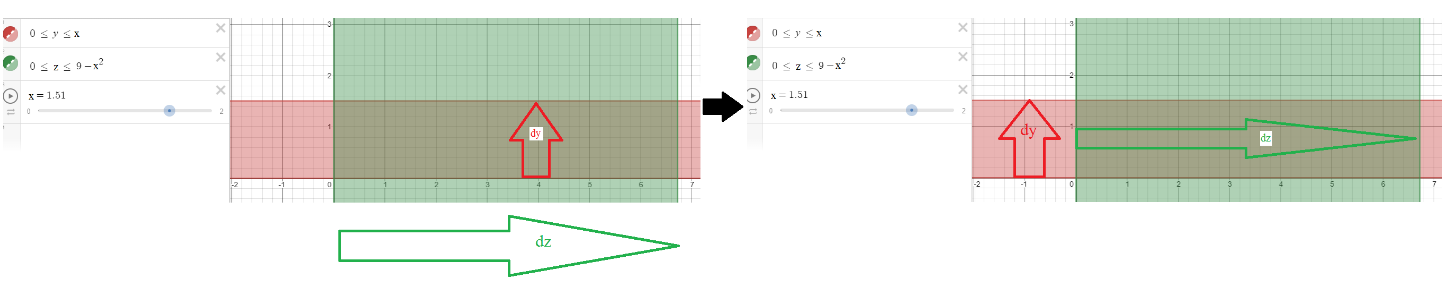





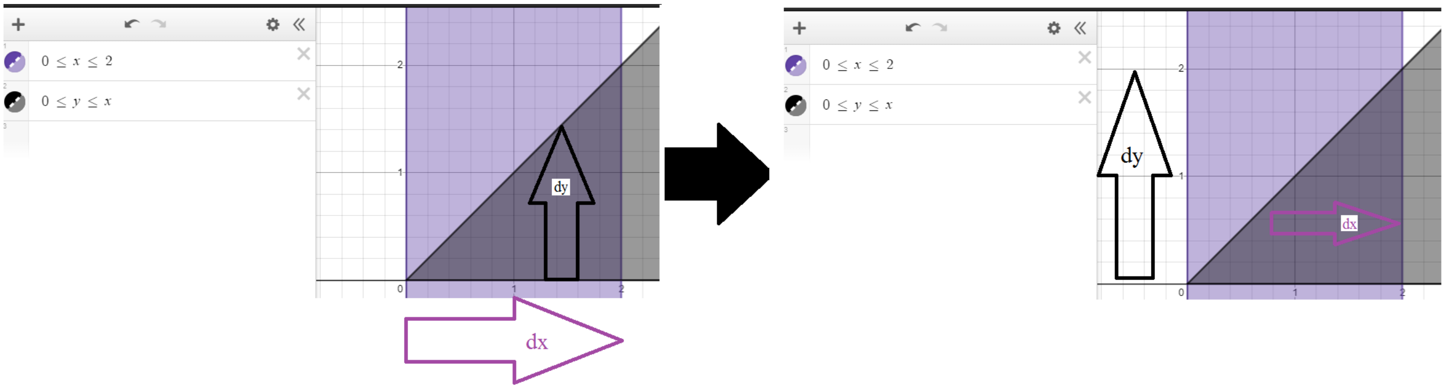

(13)
$$\boxed{\int_0^2 \int_y^2 \int_0^{9-x^2} f(x,y,z)dzdxdy}.$$

**Second Solution:** In order to avoid drawing and thinking about 3-dimensional regions, we will perform 2 separate changes of order. We will first change the order from $dydzdx$ to $dzdydx$, and then we will change the order from $dzdydx$ to $dzdxdy$.



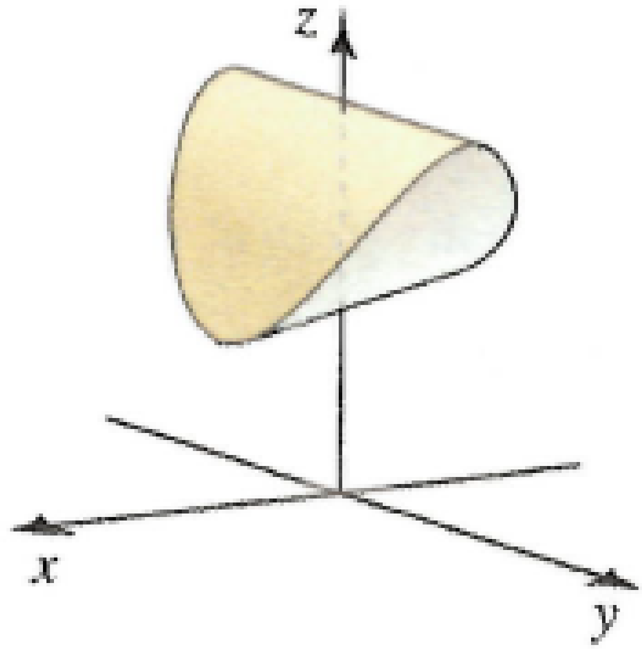



$$(14)\qquad \int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)\,dy\,dz\,dx = \int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)\,dz\,dy\,dx$$

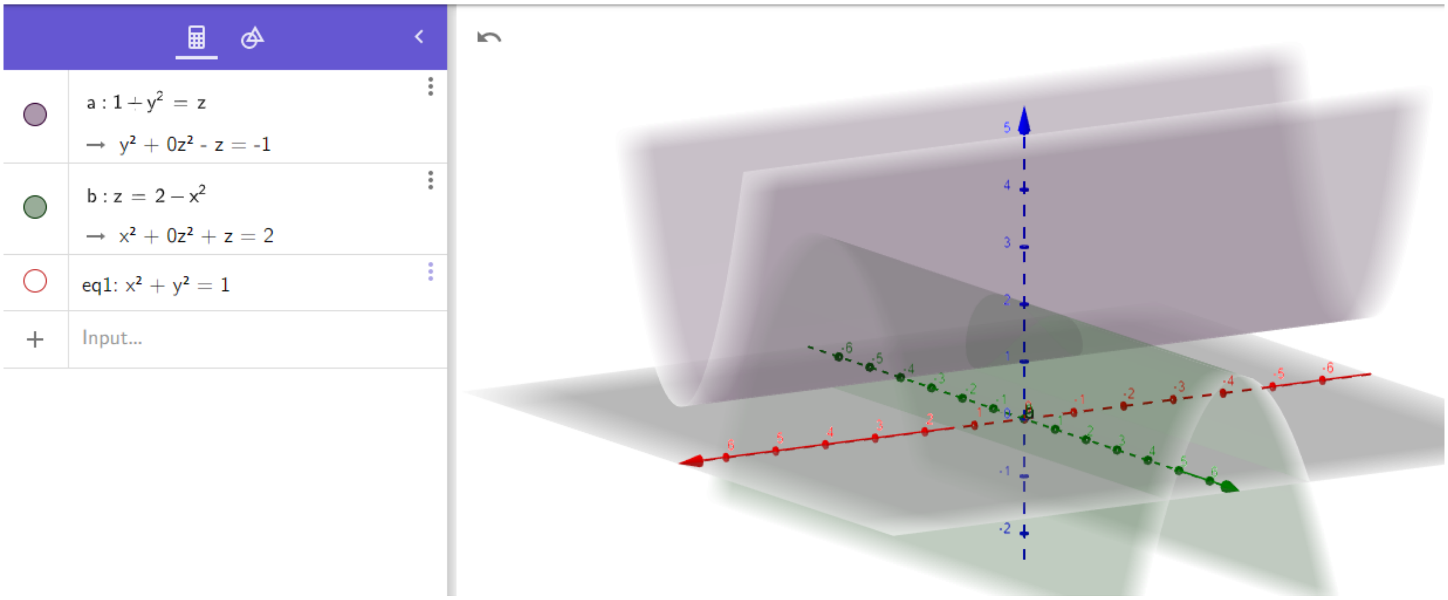


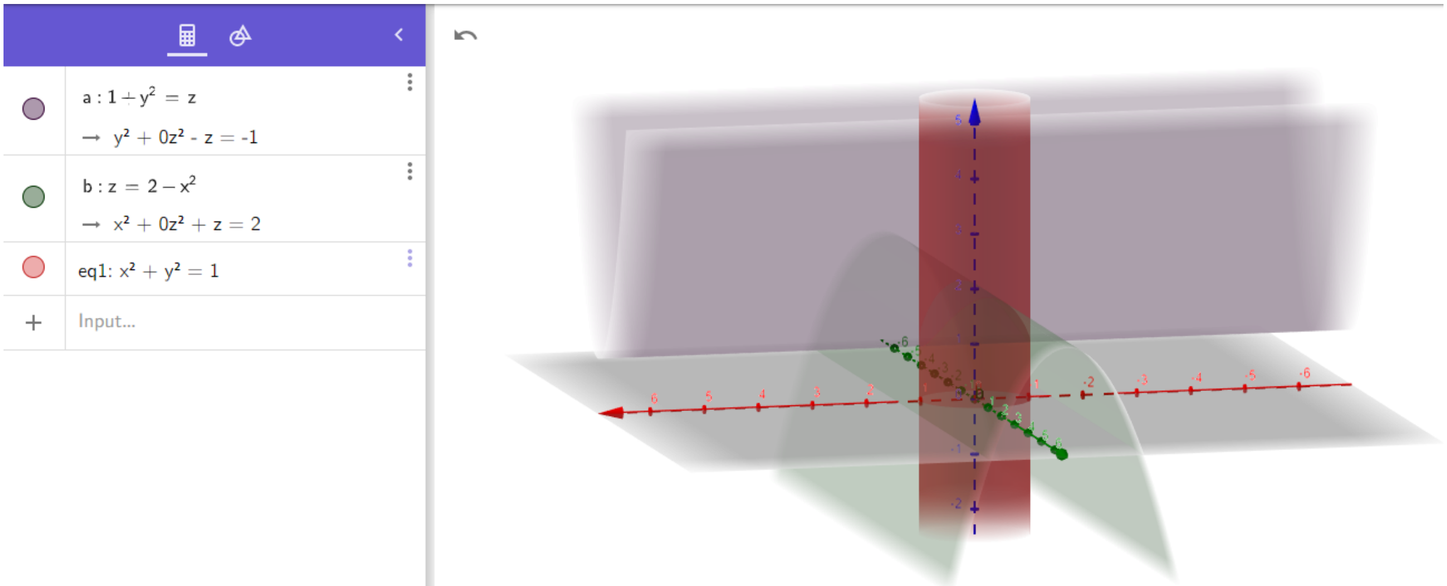

$$(15)\qquad \int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)\,dz\,dy\,dx = \boxed{\int_0^2 \int_y^2 \int_0^{9-x^2} f(x,y,z)\,dz\,dx\,dy}.$$

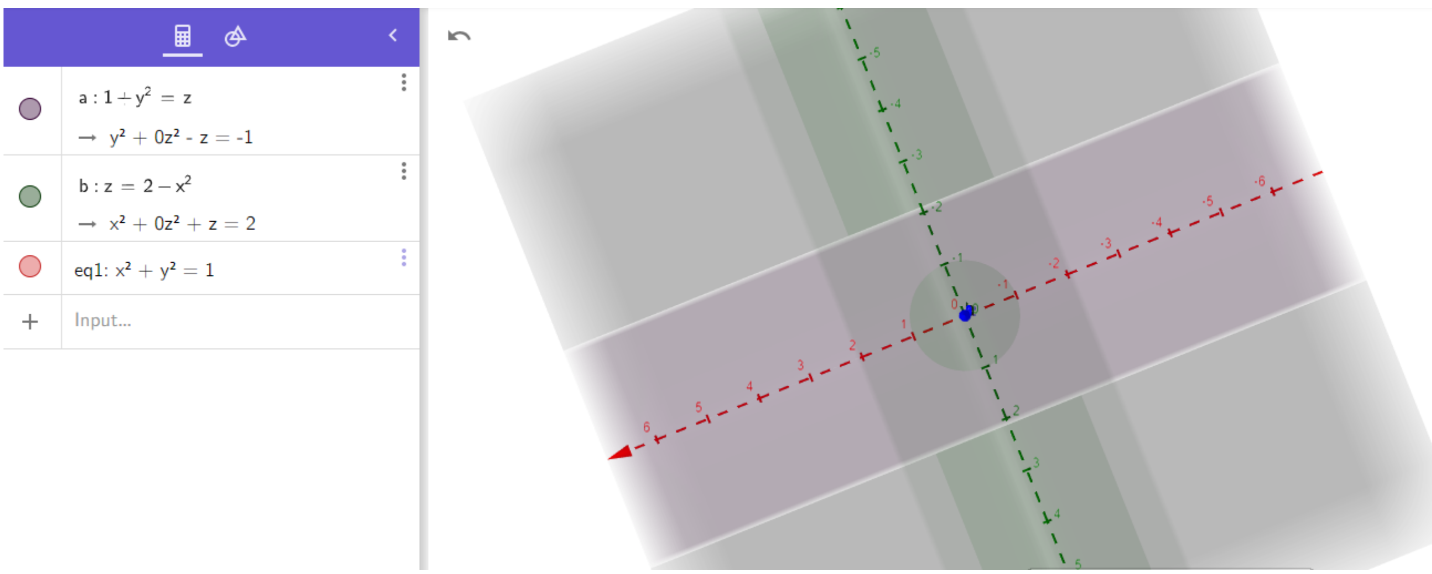
**Review Problem 2.34:** Find the volume of the solid $S$ that is bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$.




**Solution:** $S$ is a 3 dimensional solid that is defined as the region inbetween 2 surfaces. First, we find the intersection $I$ of $z = y^2 + 1$ and $z = 2 - x^2$ to satisfy $y^2 + 1 = 2 - x^2$ or $x^2 + y^2 = 1$.






It follows that the $(x, y)$-coordinates of $I$ are the circle of radius 1 centered at the origin. Note that the intersection $I$ is not itself a circle since the $z$-coordinate is not constant on the intersection. NThankfully, for the purposes of calculating the volume of $S$, we only need to know the projection $R$ of $I$ onto the $xy$-plane (along with the interior of the projection), which is the same as knowing the the $(x, y)$-coordinates of $I$.

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bottom}})dA \tag{16}$$

$$= \int_0^{2\pi} \int_0^1 \left((2 - (r\cos(\theta))^2) - ((r\sin(\theta))^2 + 1)\right) r dr d\theta \tag{17}$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) r dr d\theta \tag{18}$$

$$= \int_0^1 \int_0^{2\pi} \left(r - r^3\right) d\theta dr = \int_0^{\sqrt{3}} \left(r\theta - r^3\theta\right)\Big|_{\theta=0}^{2\pi} dr \tag{19}$$

$$= \int_0^1 2\pi \left(r - r^3\right) dr = 2\pi \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1 = \boxed{\frac{\pi}{2}}. \tag{20}$$

**Remark:** We could have also calculated the volume by using a triple integral in cylindrical coordinates as follows.

$$\text{Volume}(S) = \iiint_S 1 dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2\sin^2(\theta)+1}^{2-r^2\cos^2(\theta)} r dz dr d\theta = \boxed{\pi}. \tag{21}$$