Math 2177 Recitation
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(Pg 41) Definition: The partial derivative of f $f(x,y)$ with respect to x at the point (a,b) is

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b)-f(a,b)}{h}.$$

The partial derivative of f with respect to y at the point (a,b) is (Provided that the limits exist)

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h}.$$

Ex 1.4.45: Calculate f_x, f_y, and f_z for $f(x,y,z) = xy+xz+yz$.

Solution: $f_x(x,y,z) = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y,z)$

$= \frac{\partial}{\partial x}(xy+xz+yz) = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial x}(yz)$

$= y\frac{\partial}{\partial x}(x) + z\frac{\partial}{\partial x}(x) + 0 = y \cdot 1 + z \cdot 1 = y+z.$

$f_y(x,y,z) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y,z) =$

$\frac{\partial}{\partial y}(xy+xz+yz) = \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial y}(yz)$

$= x\frac{\partial}{\partial y}(y) + 0 + z\frac{\partial}{\partial y}(y) = x \cdot 1 + z \cdot 1 = x+z.$

Pg 1 Similarly, $f_z(x,y,z) = x+y.$ $\square$

Ex 1.4.52: Find $g_w, g_x, g_y,$ and g_z for

$g(w,x,y,z) = \cos(w+x)\sin(y-z)$.

Solution: $g_w(w,x,y,z) = \frac{\partial g}{\partial w} = \frac{\partial}{\partial w} g(w,x,y,z)$

$= \frac{\partial}{\partial w} \cos(w+x)\sin(y-z) = \sin(y-z)\frac{\partial}{\partial w}\cos(w+x)$

Chain rule → $= \sin(y-z)\left(\frac{\partial}{\partial t}\cos(t)\Big|_{t=w+x}\right)\frac{\partial}{\partial w}(w+x)$

$= \sin(y-z)(-\sin(w+x))\cdot 1 = -\sin(y-z)\sin(w+x)$.

Similarly, $g_x(w,x,y,z) = -\sin(y-z)\sin(w+x)$.

$g_y(w,x,y,z) = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y} g(w,x,y,z)$

$= \frac{\partial}{\partial y}\cos(w+x)\sin(y-z) = \cos(w+x)\frac{\partial}{\partial y}\sin(y-z)$

Chain rule → $= \cos(w+x)\left(\frac{\partial}{\partial t}\sin(t)\Big|_{t=y-z}\right)\frac{\partial}{\partial y}(y-z)$

$= \cos(w+x)\cdot\cos(y-z)\cdot 1 = \cos(w+x)\cos(y-z)$.

Similarly, $g_z(w,x,y,z) = -\cos(w+x)\cos(y-z)$.

- Partial derivatives describe the rate of change when only one variable changes and the others are held constant.

(See picture 1)

(Pg 62) (Pg 46-47)

Definition: Let f be differentiable at (a,b) and let $\vec{u} = \langle u_1, u_2 \rangle$ be a <u>unit vector</u> ($\|\vec{u}\|=1$) in the xy-plane. The directional derivative of f at (a,b) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h},$$

provided the limit exists.

$D_{\langle 1,0 \rangle} f(a,b) = f_x(a,b)$

$D_{\langle 0,1 \rangle} f(a,b) = f_y(a,b)$

Theorem: Let f be differentiable at (a,b) and let $\vec{u} = \langle u_1, u_2 \rangle$ be a <u>unit vector</u> in the xy-plane. We may calculate $D_{\vec{u}} f(a,b)$ by

$$D_{\vec{u}} f(a,b) = \underbrace{\langle f_x(a,b), f_y(a,b) \rangle}_{\vec{\nabla} f(a,b)} \cdot \langle u_1, u_2 \rangle.$$

Definition: Let f be differentiable at (a,b). The gradient vector of f at (a,b) is

$$\vec{\nabla} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle.$$

Example: For $f(x,y,z) = xy + xz + yz$, we have $\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

$= \langle y+z, x+z, x+y \rangle.$

(Pg 65)

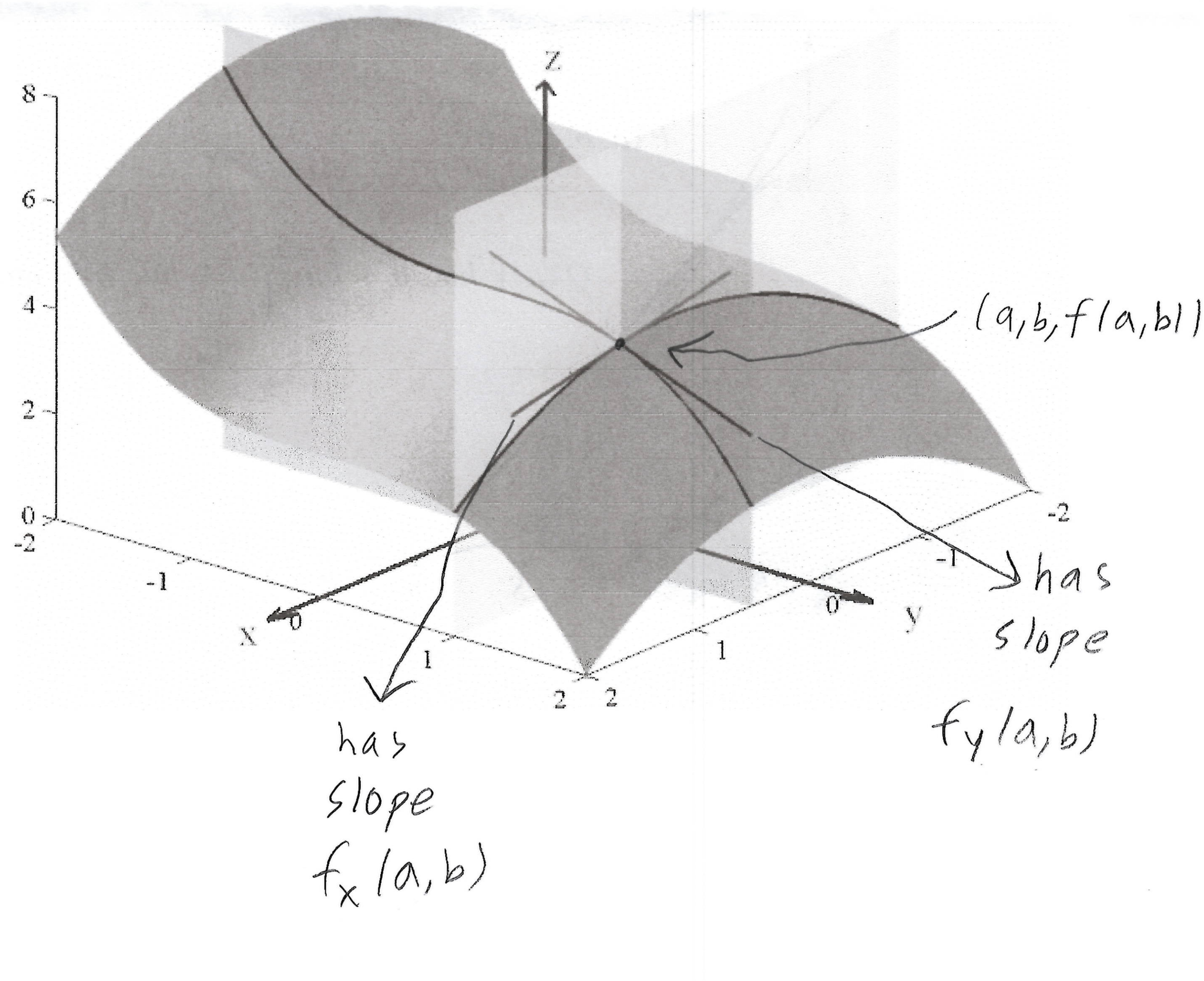
Theorem: Let f be differentiable at (a,b) with $\vec{\nabla} f(a,b) \neq \vec{0}$ (where $\vec{0} = (0,0)$).

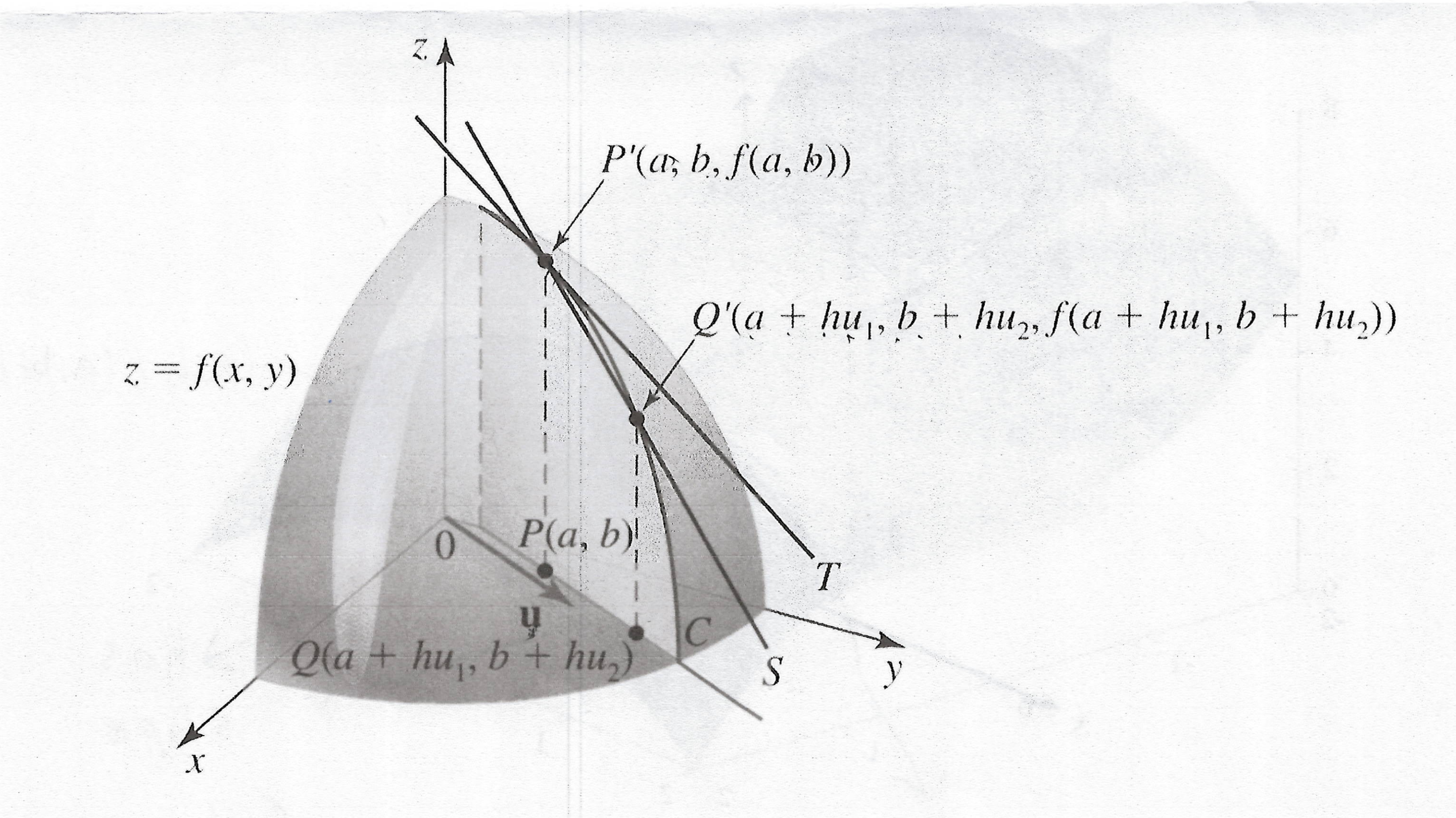
1) f has maximum rate of increase at (a,b) in the direction of the gradient $\vec{\nabla} f(a,b)$. The rate of change in this direction is $|\vec{\nabla} f(a,b)|$.

2) f has maximum rate of decrease at (a,b) in the direction of $-\vec{\nabla} f(a,b)$. The rate of change in this direction is $-|\vec{\nabla} f(a,b)|$.

3) The directional derivative is 0 in any direction orthogonal to $\vec{\nabla} f(a,b)$.

$z = f(x,y)$




z
P'(a, b, f(a, b))
Q'(a + hu$_1$, b + hu$_2$, f(a + hu$_1$, b + hu$_2$))
z = f(x, y)
0
P(a, b)
T
u
C
Q(a + hu$_1$, b + hu$_2$)
S
y
x

$f: z = x^2 + y^2$ graph of $f(x,y) = x^2+y^2$

$B = (3, 0, 1)$

$A = (1, 0, 1)$

$u = \text{Vector}(A, B)$

$\rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

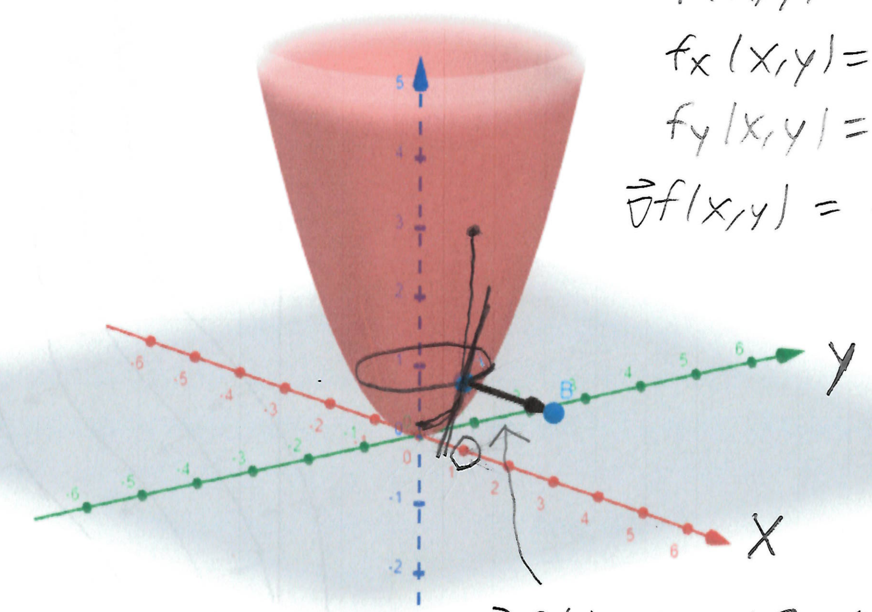
Input...

$f(x,y) = x^2+y^2$

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

$\vec{\nabla} f(x,y) = \langle 2x, 2y \rangle$

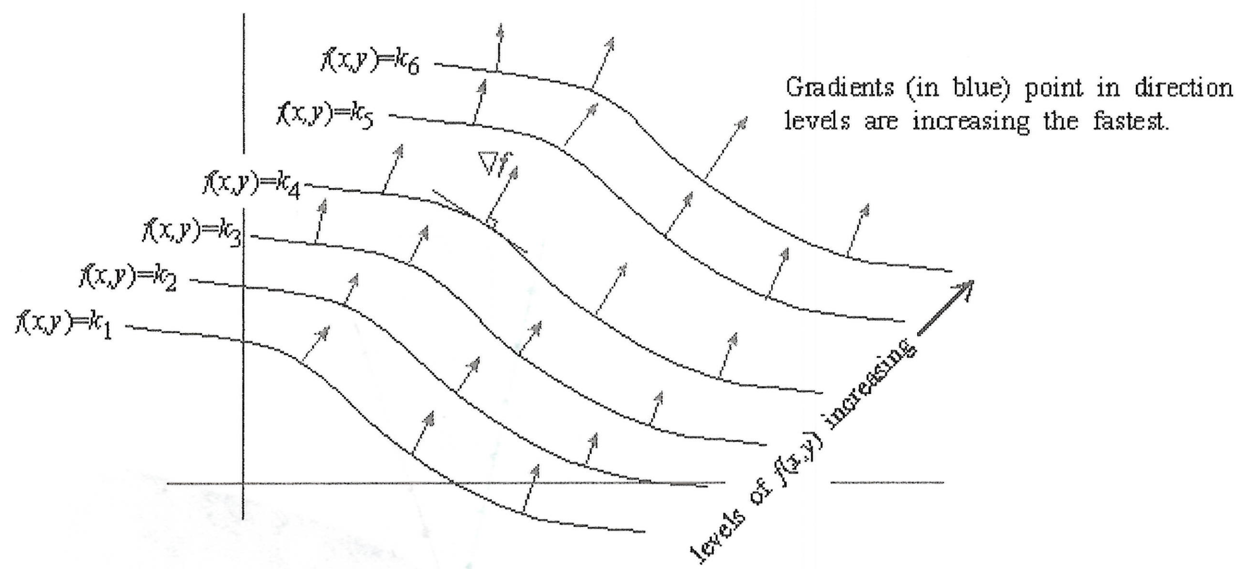


$\vec{\nabla} f(1,0) = \langle 2, 0 \rangle$

$\vec{\nabla} f(1,0)$ has direction

$$\frac{\vec{\nabla} f(1,0)}{\|\vec{\nabla} f(1,0)\|} = \frac{\langle 2,0 \rangle}{\sqrt{2^2+0^2}} = \langle 1,0 \rangle$$

($\vec{u}$ has direction $\dfrac{\vec{u}}{\|\vec{u}\|}$ if $\|\vec{u}\| \neq 0$)


$f(x,y)=k_6$
$f(x,y)=k_5$
$f(x,y)=k_4$
$f(x,y)=k_3$
$f(x,y)=k_2$
$f(x,y)=k_1$
∇f
Gradients (in blue) point in direction levels are increasing the fastest.
levels of $f(x,y)$ increasing