Jacobian Determinant of a Transformation of Two Variables.

Given a transformation $T: x = g(u,v),\ y = h(u,v)$, where g and h are differentiable on a region of the uv-plane, the Jacobian determinant (or Jacobian) of T is

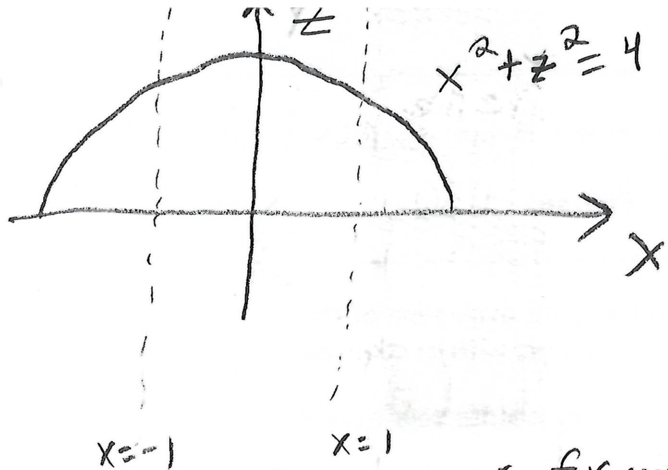
$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}.$$

Change of Variables for Double Integrals

Let $T: x = g(u,v),\ y = h(u,v)$ be a transformation that maps a closed bounded region S in the uv-plane onto a region R in the xy-plane. assume that T is one-to-one on the interior of S and that g and h have continuous first partial derivatives there. If f is continuous, then

$$\iint_R f(x,y)\, dA = \iint_S f(g(u,v), h(u,v))\,|J(u,v)|\, dA.$$

2.5.70)



also comes from $r=1$

The xz-plane cross-section of the solid shows that cylindrical coordinates will be useful.

$$V = \int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^{2} r\sqrt{4-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{(4-r^2)^{\frac{3}{2}}}{3}\Big|_{r=1}^{2}\,d\theta = \int_0^{2\pi}\frac{3^{\frac{3}{2}}}{3}\,d\theta = \frac{2\pi\,3^{3/2}}{3}$$

$$= 2\cdot 3^{\frac{1}{2}}\pi = \boxed{2\sqrt{3}\,\pi}$$

2.5.62) $I = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\sec\varphi} g(\rho,\varphi,\theta)\,\rho^2\,d\rho\,d\varphi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^{4\sec\varphi} g(\rho,\varphi,\theta)\,\rho^2\,d\rho\,d\theta\,d\varphi =$$

$$\int_0^{\pi/2}\int_0^{4\sec\varphi}\int_0^{2\pi} g(\rho,\varphi,\theta)\,\rho^2\,d\theta\,d\rho\,d\varphi.$$

Since θ has no relationship with ρ or φ, changing the order of integration is trivial. It is harder to $d\theta\,d\varphi\,d\rho$, so let's do that. $\rho = 4\sec\varphi = \frac{4}{\cos\varphi} \rightarrow 4 = \rho\cos\varphi$ $\rightarrow \cos\varphi = \frac{4}{\rho} \rightarrow \varphi = \cos^{-1}\left(\frac{4}{\rho}\right)$. also, as $\varphi \rightarrow \frac{\pi}{2}^-, 4\sec\varphi \rightarrow \infty$, so

$$I = \int_0^{\infty}\int_0^{\cos^{-1}\left(\frac{4}{\rho}\right)}\int_0^{2\pi} g(\rho,\varphi,\theta)\,\rho^2\,d\theta\,d\varphi\,d\rho$$

also, let us consider converting to Cartesian coordinates.

$\rho = 4\sec(\varphi) \Rightarrow 4 = \rho\cos(\varphi) = z$, so

$$I = \int_0^{\infty}\int_0^{\infty}\int_0^{4} g'(x,y,z)\,dz\,dy\,dx.$$

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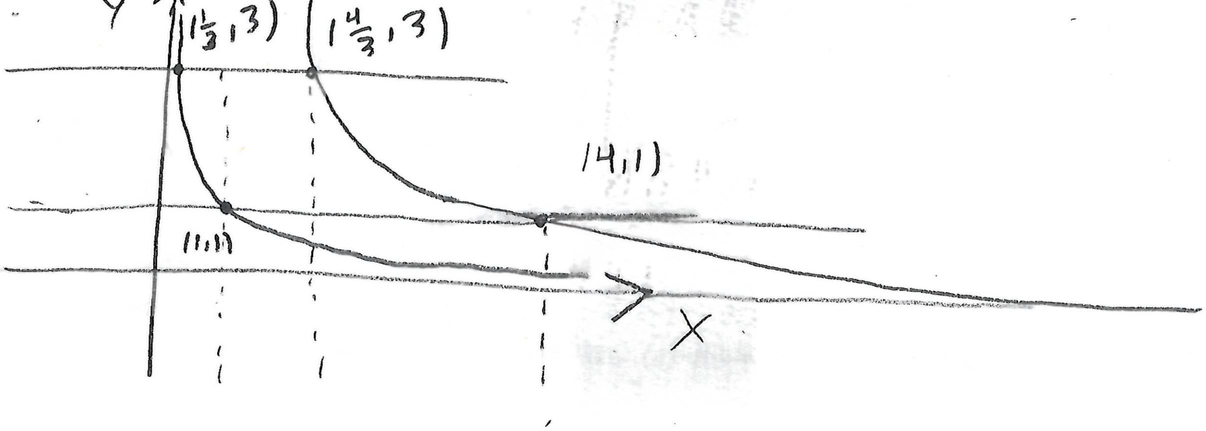
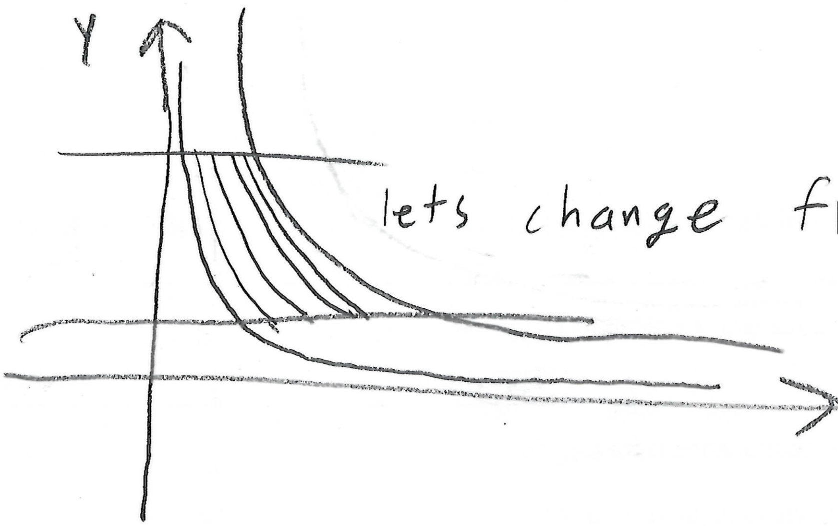
35)





lets change from x, y to u, y with $u = xy$, so $1 \le u \le 4$, $1 \le y \le 3$, and we are integrating over a rectangle.

Let $u = xy$, $v = y$, so $x = \frac{u}{y} = \frac{u}{v}$, $y = v$.

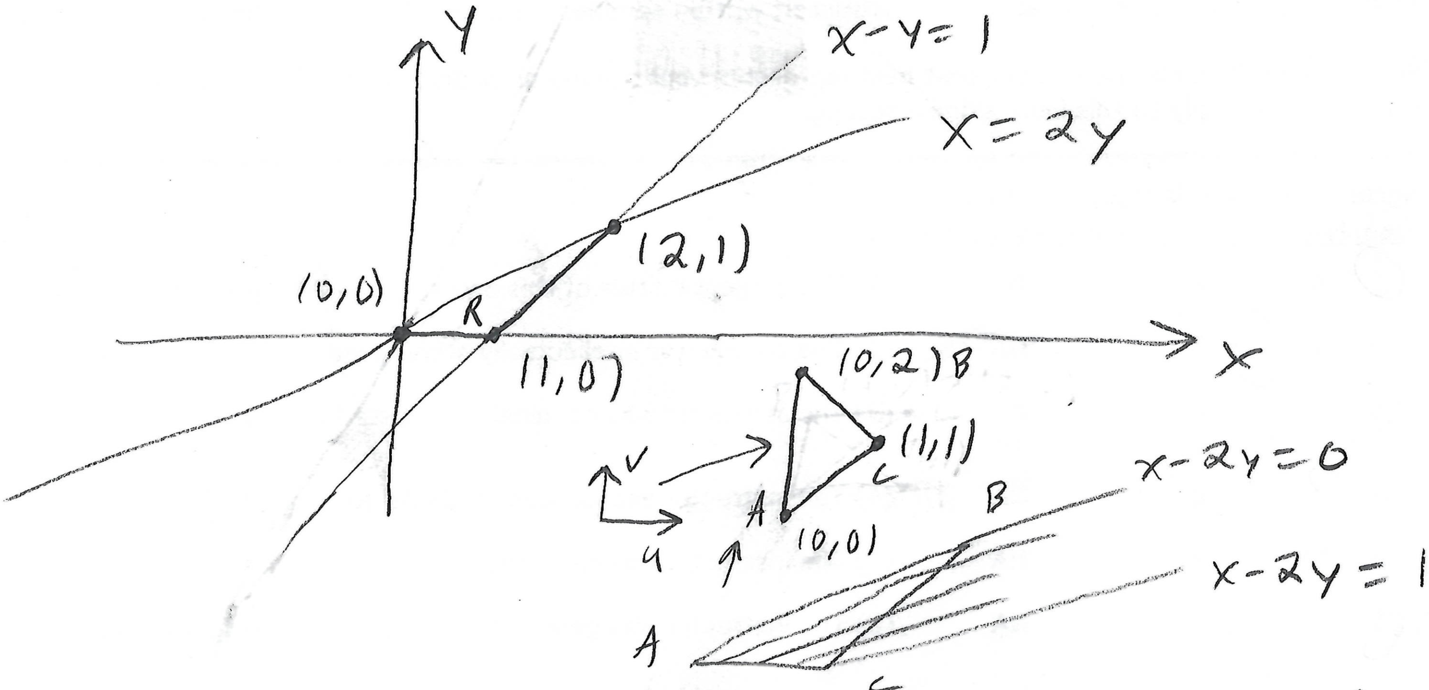
we see that $g(u,v) = \frac{u}{v}$, $h(u,v) = v$, and that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}, \text{ so}$$

$$\iint_R xy\, dA = \iint_S u \cdot |J(u,v)|\, dA = \int_1^3 \int_1^4 \frac{u}{v}\, du\, dv$$

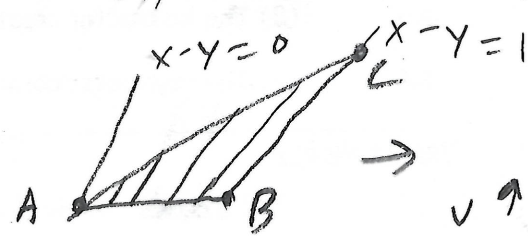
$$= \int_1^3 \frac{u^2}{2v}\Big|_{u=1}^{4} dv = \int_1^3 \frac{15}{2v}\,dv = \frac{15}{2}\ln(v)\Big|_1^3 = \boxed{\frac{15\ln(3)}{2}}$$

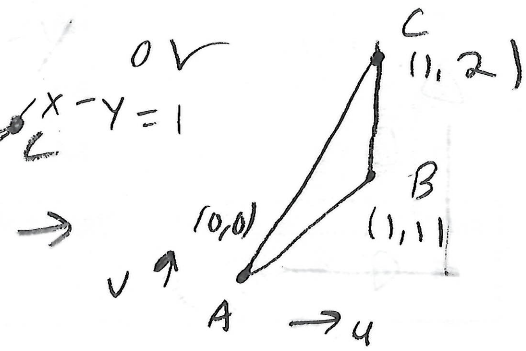
2.7.36)



Let $u = x-2y,\ v = x$ or

$u = x-y,\ v = x$





Let's start with $u = x-2y,\ v = x$, so

$x = v,\ y = \frac{x-u}{2} = \frac{v-u}{2}$, hence

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2},\ \text{hence}$$

$$\iint_R (x-y)\sqrt{x-2y}\,dA = \iint_S \left(\tfrac{1}{2}v+\tfrac{1}{2}u\right)\sqrt{u}\cdot|J(u,v)|\,dA$$

$$= \int_0^1 \int_u^{2-u} \left(\tfrac{1}{2}v\sqrt{u}+\tfrac{1}{2}u^{\frac{3}{2}}\right)dv\,du \cancel{= \int_0^1 \left(\frac{v^2\sqrt{u}}{4}+\frac{1}{2}u^{\frac{3}{2}}v\right)\Big|_{v=0}^{1} du}$$

$$\cancel{= \int_0^1 \left(\frac{\sqrt{u}}{4}+\frac{1}{2}u^{\frac{3}{2}}\right)du = \frac{u^{\frac{3}{2}}}{6}+\frac{u^{\frac{5}{2}}}{5}\Big|_0^1 = \frac{1}{6}+\frac{1}{5} = \boxed{\frac{11}{30}}}$$

Alternatively, had we done $u=x-y, v=x$, then $x=v$, $y=x-u=v-u$, so

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1, \text{ hence}$$

$$\iint_R (x-y)\sqrt{x-2y}\,dA = \iint_S u\sqrt{2u-v}\,|J(u,v)|\,dA$$

$$= \int_0^1\int_u^{2u} u\sqrt{2u-v}\,dv\,du \cancel{= \int_0^1 \frac{-2u(2u-v)^{\frac{3}{2}}}{3}\Big|_{v=0}^{1} du =}$$

We leave the final calculations to the reader.