26, 21, 49 → 2.1

1.9) 16, 22, 27

Method of Lagrange Multipliers in Three Variables.
Let f and g be differentiable on a region R^3 with $\nabla g(x,y,z) \neq 0$ on the surface $g(x,y,z)=0$. To locate the maximum and minimum values of f subject to the constraint $g(x,y,z)=0$, carry out the following steps.

1) Find the values x, y, z, and λ that satisfy the equations

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ and $g(x,y,z)=0$.

2) Among the points (x,y,z) found in Step 1, select the largest and smallest corresponding function values. These are the ~~local~~ minimums and maximums of f subject to $g(x,y,z)=0$ if $g(x,y,z)=0$ is bounded.

16) $f(x,y,z) = xyz$ subject to $x^2+2y^2+4z^2=9$

$x^2+2y^2+4z^2-9=0$

$g(x,y,z) = x^2+2y^2+4z^2-9$

$\frac{\partial f}{\partial x} = yz, \quad \frac{\partial f}{\partial y} = xz, \quad \frac{\partial f}{\partial z} = xy$

$\frac{\partial g}{\partial x} = 2x, \quad \frac{\partial g}{\partial y} = 4y, \quad \frac{\partial g}{\partial z} = 8z$

$\nabla f(x,y,z) = \langle yz, xz, xy \rangle$

$\nabla g(x,y,z) = \langle 2x, 4y, 8z \rangle$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle yz, xz, xy \rangle = \langle 2x\lambda, 4y\lambda, 8z\lambda \rangle$

(1) $yz = 2x\lambda$

(2) $xz = 4y\lambda$

(3) $xy = 8z\lambda$

(1)&(2) $4y^2z\lambda = 2x^2z\lambda$

$4y^2z\lambda - 2x^2z\lambda = 0$

$2z\lambda(2y^2 - x^2) = 0$

Pg 1

$z=0, \lambda=0$, or $2y^2-x^2=0$. If $z=0$, then

$0=2x\lambda \rightarrow x=0$ or $\lambda=0$.

$0=4y\lambda$ If $x=0$, then $9=x^2+2y^2+4z^2=0+2y^2+0=2y^2$

$xy=0$ $\rightarrow y=\pm\frac{3}{\sqrt{2}} \rightarrow \lambda=0 \rightarrow$

$(0, \frac{3}{\sqrt{2}}, 0), (0, \frac{-3}{\sqrt{2}}, 0)$.

$(xy=0)$

If $\lambda=0$ and $x\neq 0$, then $y=0$, so

$9=x^2+0+0=x^2 \rightarrow x=\pm 3 \rightarrow (3,0,0), (-3,0,0)$

If $z\neq 0$ and $\lambda=0$, then

$yz=0 \rightarrow y=0 \rightarrow 9=0+0+4z^2 \rightarrow z=\pm\frac{3}{2} \rightarrow (0,0,\frac{3}{2}), (0,0,\frac{-3}{2})$

$xz=0 \rightarrow x=0$

$xy=0$

If $z\neq 0$, $\lambda\neq 0$ and $2y^2-x^2=0$, so $x=\pm\sqrt{2}y$

Similarly, $2y^2=4z^2$, so $x^2=2y^2=4z^2=3$.

18) $f(x,y,z) = x - z$ subject to $x^2+y^2+z^2-y=2$

1.9.18)

$\nabla f(x,y,z) = \langle 1,0,-1\rangle$, $\nabla g(x,y,z) = \langle 2x, 2y-1, 2z\rangle$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$\langle 1,0,-1\rangle = \langle 2x\lambda, (2y-1)\lambda, 2z\lambda\rangle$

(1) $1 = 2x\lambda$

(2) $0 = (2y-1)\lambda \to 2y-1=0$ or $\lambda = 0$.

$\lambda = 0 \to 1 = 0$, which is a contradiction, so $\lambda \neq 0$.

(3) $-1 = 2z\lambda$

if $2y-1=0 \to y = \frac{1}{2}$.

from multiplying (1) and (3) gives

$2z\lambda = -2x\lambda \to 2z\lambda + 2x\lambda = 0 \to 2\lambda(z+x) = 0$. Since $\lambda \neq 0$,

$z+x=0$, so $z=-x$.

$2 = x^2+y^2-y+z^2 = x^2+\frac{1}{4}-\frac{1}{2}+(-x)^2 = 2x^2-\frac{1}{4} \to$

$2x^2 = 2+\frac{1}{4} = \frac{9}{4} \to x^2 = \frac{9}{8} \to x = \pm\frac{3}{2\sqrt{2}}$,

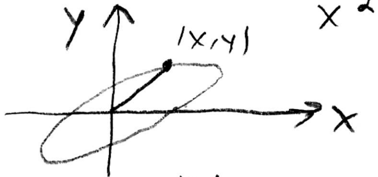
$\left(\frac{3}{2\sqrt{2}}, \frac{1}{2}, \frac{-3}{2\sqrt{2}}\right), \left(\frac{-3}{2\sqrt{2}}, \frac{1}{2}, \frac{3}{2\sqrt{2}}\right)$ $\vec{\nabla} g(x,y,z) \neq 0$.

$x - z = \frac{3}{\sqrt{2}}$ max

$x - z = \frac{-3}{\sqrt{2}}$ min

27)
1.9.27)

$x^2+xy+y^2=1$



distance $=\sqrt{x^2+y^2}$

(Because $\sqrt{z}$ is an increasing function)

The minimum and maximum of $\sqrt{x^2+y^2}$ is the same as that of x^2+y^2, so we will use $f(x,y)=x^2+y^2$ and $g(x,y)=x^2+xy+y^2-1$

$\nabla f(x,y) = \langle 2x, 2y\rangle$, $\nabla g(x,y) = \langle 2x+y, 2y+x\rangle$

$\nabla f(x,y) = \lambda \nabla g(x,y)$

(1) $2x = \lambda(2x+y)$

(2) $2y = \lambda(2y+x)$

Cross multiplying (1) and (2) gives

$2x\lambda(2y+x) = 2y\lambda(2x+y)$

$\lambda(2x(2y+x) - 2y(2x+y)) = 0$

$\lambda(2x^2-2y^2)=0$, so

$\lambda=0$ or $2x^2-2y^2=0$.

If $\lambda=0$, then $x=y=0$, but $g(0,0)\neq 0$, so $(0,0)$ does not satisfy the constraint and $\lambda\neq 0$.

If $2x^2-2y^2=0$, then $2x^2=2y^2 \to x^2=y^2 \to x=\pm y$.

subcase 1: $x=y \Rightarrow$ $0=g(x,x)=3x^2-1 \Rightarrow x=\pm\sqrt{\frac{1}{3}} \Rightarrow$

$2x=3\lambda x \Rightarrow \lambda=\frac{2}{3}$

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

subcase 2: $x=-y \Rightarrow$ $2x=\lambda x$

$y=-x$ $\qquad -2x=\lambda(-x) \Rightarrow \lambda=2$

$g(x,-x)=0 \Rightarrow x^2-1=0 \Rightarrow x=\pm 1 \to (1,-1), (-1,1)$

Now we plug back into $\sqrt{x^2+y^2}$.

max is $\sqrt{2}$, min is $\sqrt{\frac{2}{3}}$.

2.1.49) $f(x,y) = 24-3x-4y$, $R = \{(x,y) \mid -1 \le x \le 3, 0 \le y \le 2\}$

$$\iint_R f(x,y)\,dA = \int_{-1}^{3}\int_{0}^{2} (24-3x-4y)\,dy\,dx$$

$$= \int_{-1}^{3} 24y-3xy-2y^2 \Big|_{y=0}^{2} dx = \int_{-1}^{3} (48-6x-8)\,dx =$$

$$\int_{-1}^{3} (40-6x)\,dx = 40x-3x^2 \Big|_{x=-1}^{3} = (120-27)-(-40-3)$$

$$= 93+43 = 136.$$

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26) $\iint_R y\cos(xy)\,dA$, $R = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le \pi/3\}$

$$\iint_R y\cos(xy)\,dA = \int_0^1 \int_0^{\pi/3} \underset{\substack{\downarrow \\ \text{hard}}}{y\cos(xy)\,dy\,dx} = \int_0^{\pi/3}\int_0^1 \underset{\substack{\downarrow \\ \text{doable}}}{y\cos(xy)\,dx\,dy}$$

$u = xy$

$\frac{\partial u}{\partial x} = y\,dx$

$$\int_0^{\pi/3}\int_0^1 y\cos(xy)\,dx\,dy = \int_0^{\pi/3}\int_{x=0}^{x=1} \cos(u)\,du\,dy$$

$$= \int_0^{\pi/3} \sin(u)\Big|_{x=0}^{x=1} dy = \int_0^{\pi/3} \sin(xy)\Big|_{x=0}^{x=1} dy$$

$$= \int_0^{\pi/3} \sin(y)\,dy = -\cos(y)\Big|_{y=0}^{\pi/3} = -\frac{1}{2}+1 = \frac{1}{2}.$$

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