Theorem 1.14: If f has a local maximum or minimum value at (a,b) and the partial derivatives f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$.

Definition: a function f has a <u>local maximum value</u> at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f in some open disk centered at (a,b). a function f has a <u>local minimum value</u> at (a,b) if $f(x,y) \ge f(a,b)$ for all (x,y) in the domain of f in some open disk centered at (a,b). Local maximum and local minimum values are also called <u>local extreme values</u> or <u>local extrema</u>.

Definition: An interior point is called a <u>critical point</u> of f if either

1) $f_x(a,b) = f_y(a,b) = 0$, or
2) at least one of the partial derivatives f_x and f_y does not exist at (a,b).

Definition: Consider a function f that is differentiable at a critical point (a,b). Then f has a saddle point at (a,b) if, in every open disk centered at (a,b), there are points (x,y) for which $f(x,y) > f(a,b)$ and points for which $f(x,y) < f(a,b)$.

Theorem 1.15 (The Second Derivative Test)

Suppose that the second partial derivatives of f are continuous throughout an open disk centered at the point (a,b), where $f_x(a,b) = f_y(a,b) = 0$. Let $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - (f_{xy}(x,y))^2$

1) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum value at (a,b).

2) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum value at (a,b)

3) If $D(a,b) < 0$, then f has a saddle point at (a,b).

4) If $D(a,b) = 0$, then the test is inconclusive.

~~P23)~~ i) Find the critical points.

ii) Use the second derivative test to determine if the point is a local minimum, local maximum, or a saddle point.

P23) $f(x,y) = x^4 + 2y^2 - 4xy$

$f_x(x,y) = 4x^3 + 0 - 4y = 4x^3 - 4y$, $f_{xx}(x,y) = 12x^2$

$f_{xy}(x,y) = 0 - 4 = -4$

$f_y(x,y) = 0 + 4y - 4x = 4y - 4x$, $f_{yy}(x,y) = 4$

To find the critical points, we have to solve the system.

$f_x(x,y) = 0$

$f_y(x,y) = 0$

$4x^3 - 4y = 0$

$4y - 4x = 0$

$x^3 = y$

$x = y$

$x^3 = x$

$x^3 - x = 0$

$x(x^2 - 1) = 0$

$x(x-1)(x+1) = 0$

$x = 0, 1, -1$, so the critical points are $\{(-1,-1), (0,0), (1,1)\}$.

Recalling that

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$,

we see that

$D(1,1) = 12 \cdot 1 \cdot 4 - (-4)^2 = 48 - 16 = 32 > 0$

$f_{xx}(1,1) = 12 > 0$, so $(1,1)$ is a local min. of f.

$D(0,0) = 12 \cdot 0 \cdot 4 - (-4)^2 = -16 < 0$, so $(0,0)$ is a saddle point of f.

$D(-1,-1) = 12 \cdot (-1)^2 \cdot 4 - (-4)^2 = 32 > 0$ $(-1,-1)$ is

$f_{xx}(-1,-1) = 12 > 0$

24) $f(x,y) = xy e^{-x-y}$ $= \frac{\partial}{\partial x}(xye^{-x-y})$

$$f_x(x,y) = \frac{\partial}{\partial x}(xy)e^{-x-y} + xy\frac{\partial}{\partial x}(e^{-x-y})$$

$$= ye^{-x-y} + xy\, e^{-x-y}\frac{\partial}{\partial x}(-x-y) \quad \text{chain rule}$$

$$= ye^{-x-y} - xye^{-x-y}$$

$$f_{xx}(x,y) = y\frac{\partial}{\partial x}(e^{-x-y}) - \frac{\partial}{\partial x}(xye^{-x-y})$$

$$= y(-e^{-x-y}) - (ye^{-x-y} - xye^{-x-y})$$

$$= -2ye^{-x-y} + xye^{-x-y}$$

$$f_y(x,y) = xe^{-x-y} - xye^{-x-y}$$

$$f_{yy}(x,y) = -2xe^{-x-y} + xye^{-x-y} \quad \text{by symmetry}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}(y)e^{-x-y} + y\frac{\partial}{\partial y}(e^{-x-y}) - \frac{\partial}{\partial y}(xye^{-x-y})$$

$$= e^{-x-y} + y(-e^{-x-y}) - (xe^{-x-y} - xye^{-x-y})$$

$$= e^{-x-y} - ye^{-x-y} - xe^{-x-y} + xye^{-x-y}$$

To find critical points, we solve the system

$f_x(x,y) = 0$, $f_y(x,y) = 0$ / $ye^{-x-y} - xye^{-x-y} = 0$, $xe^{-x-y} - xye^{-x-y} = 0$ $e^{-x-y} \neq 0$, so

$y - xy = 0$, $x - xy = 0$ / $y = xy$, $x = xy$ $\rightarrow$ $x = y \Rightarrow x = x^2$ $\rightarrow x = 0, 1 \rightarrow$ The critical points are $\{(0,0), (1,1)\}$.

$D(0,0) = 0\cdot 0 - (e^{-0-0})^2 = -1 < 0 \Rightarrow (0,0)$ is a saddle point of f.

$D(1,1) = (-2e^{-2} + e^{-2})^2 - (0)^2 = e^{-4} > 0$, $f_{xx}(1,1) = -e^{-2}$, so f

~~(1,1)~~ has a local maximum at (1,1).

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i) Show that (0,0) is a critical point.

ii) Show that the second derivative test is inconclusive at (0,0).

40) $f(x,y) = x^2y - 3$, 42) $f(x,y) = \sin(x^2y^2)$