4.6. **42.** Let A be a (2×2) matrix, and let B and C be given by

$$B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

a) If $A^T + B = C$, what is A?

b) If $A^T B = C$, what is A?

---

a) $\underset{-B}{A^T + B} = \underset{-B}{C} \rightarrow A^T = C - B$

$$A = (A^T)^T = ((C-B))^T = C^T - B^T$$

$$= \left(\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \right)^T$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

---

b) $A^TB=C$

$$(A^TB)B^{-1}=CB^{-1}$$

$$\cancel{B^{-1}(A^TB)=B^{-1}C}$$

$$\cancel{(B^{-1}A^T)B=B^{-1}C}$$

$$A^T(BB^{-1})=CB^{-1}$$

$$A^TI_2=CB^{-1}$$

$$A^T=CB^{-1}$$

$$A=(A^T)^T=(CB^{-1})^T$$

$$=(B^{-1})^TC^T=(B^T)^{-1}C^T$$

$$B=\begin{bmatrix}1&3\\1&4\end{bmatrix}$$

$\Delta = 1\cdot 4 - 1\cdot 3 = 1 \neq 0 \Rightarrow$

B has an inverse.

$$B^{-1} = \frac{1}{\Delta}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow$$

$$CB^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot 4 + 3(-1) & 2(-3) + 3\cdot 1 \\ 4\cdot 4 + 5(-1) & 4(-3) + 5\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ 11 & -7 \end{bmatrix} \Rightarrow A = (CB^{-1})^T = \begin{bmatrix} 5 & -3 \\ 11 & -7 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 11 \\ -3 & -7 \end{bmatrix}$$

In Exercises 28–33, determine conditions on the scalars so that the set of vectors is linearly dependent.

4.7. **32.** $\mathbf{v}_1 = \begin{bmatrix} a \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} b \\ 3 \end{bmatrix}$

v_1 and v_2 are linearly dependent if there exist constants c_1 and c_2 such that

$c_1v_1 + c_2v_2 + c_3v_3 = 0$

$(c_1, c_2, c_3) \neq (0,0,0)$

$c_1v_1 + c_2 v_2 = 0$, (0 vector)

and $(c_1, c_2) \neq (0, 0)$.

$$c_1 \begin{bmatrix} a \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} b \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$ac_1 + bc_2 = 0$$

$$c_1 + 3c_2 = 0 \rightarrow c_1 = -3c_2$$

$$a(-3c_2) + bc_2 = 0$$

$\rightarrow \quad (b-3a)c_2 = 0$

Case 1: $c_2 = 0$.

$c_1 = -3c_2 = -3 \cdot 0 = 0 \Rightarrow$

$(c_1, c_2) = (0,0)$, so case 1

yields no solutions.

Case 2: $c_2 \neq 0$.

$$b - 3a = \frac{(b-3a)c_2}{c_2} = \frac{0}{c_2} = 0$$

$\rightarrow \boxed{b = 3a}$ is necessary.

$\rightarrow$ If $b = 3a$ then

c_2 is free and

$c_1 = -3c_2$, so for

example $(c_1, c_2) = (-3, 1) \neq (0,0)$

satisfies

$c_1 v_1 + c_2 v_2 = 0$.

$$-3\begin{bmatrix} a \\ 1 \end{bmatrix} + \begin{bmatrix} b = 3a \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} a \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3a \\ 3 \end{bmatrix}$

are linearly dependent.

In Exercises 27–28 determine the value(s) of λ for which A has an inverse.

4.9.27. $A = \begin{bmatrix} \lambda & 4 \\ 1 & \lambda \end{bmatrix}$

$\Delta = \lambda \cdot \lambda - 1 \cdot 4 = \lambda^2 - 4$

- A is invertible if and only if $\Delta \neq 0$.
- So A has an inverse if $\lambda^2 - 4 \neq 0$.
- $0 = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$
  $\rightarrow \lambda = \pm 2$, so

  $\lambda^2 - 4 \neq 0$ if $\lambda \neq \pm 2$

$\rightarrow$ A has an inverse if $\lambda \neq \pm 2$.

60–63. Not really second-order *An equation of the form $y'' = F(t, y')$ (where F does not depend on y) can be viewed as a first-order equation in y'. It may be attacked in two steps:* **(a)** *Let $v = y'$ and solve the first-order equation $v' = F(t, v)$.* **(b)** *Having determined v, solve the first order equation $y' = v$. Use this method to find the general solution of the following equations.*

5.1. **63.** $y'' = 2t(y')^2$

$$v = y', \quad v' = y''$$

$$v' = 2t v^2$$

$$\frac{v'}{v^2} = 2t \rightarrow \frac{\frac{dv}{dt}}{v^2} = 2t$$

$$\rightarrow \quad \frac{dv}{v^2} = 2t\,dt \rightarrow \int \frac{dt}{v^2} = ???$$

$$\int \frac{dv}{v^2} = \int 2t\,dt$$

$$-\frac{1}{v} = t^2 + C$$

$$-1 = (t^2 + C)v$$

$$v = -\frac{1}{t^2 + C}$$

$$y' = -\frac{1}{t^2+c}$$

$$\frac{dy}{dt} = -\frac{1}{t^2+c}$$

$$dy = -\frac{dt}{t^2+c}$$

$$\int dy = \int -\frac{dt}{t^2+c}$$

$$y = -\int \frac{dt}{t^2+c}$$

Case 1: $c=0$.

$$y = -\int \frac{dt}{t^2} = \frac{1}{t} + c_2$$

Case 2: $c>0$. $\boxed{\frac{d}{dt}\tan^{-1}(t) = \frac{1}{t^2+1}}$

$$y = -\int \frac{dt}{t^2+c} = -\frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{t}{\sqrt{c}}\right) + c_2$$

Case 3: $c < 0$, then $c = -p$ for some $p > 0$. $\left(\begin{array}{l} p = -c, \\ c = -(-c) \end{array} \right)$

$$y = -\int \frac{dt}{t^2 + c} = -\int \frac{dt}{t^2 - p}$$

$$= -\int \frac{dt}{(t-\sqrt{p})(t+\sqrt{p})}$$

$$= -\int \frac{\left(\frac{1}{t-\sqrt{p}} - \frac{1}{t+\sqrt{p}} \right)}{2\sqrt{p}} dt$$

$$= -\frac{1}{2\sqrt{p}} \int \left(\frac{1}{t-\sqrt{p}} - \frac{1}{t+\sqrt{p}} \right) dt$$

$$= -\frac{1}{2\sqrt{p}}\left(\ln|t-\sqrt{p}| - \ln|t+\sqrt{p}|\right) + C_2$$

$$= -\frac{1}{2\sqrt{p}}\ln\left|\frac{t-\sqrt{p}}{t+\sqrt{p}}\right| + C_2$$

$$= -\frac{1}{2\sqrt{-c}}\ln\left|\frac{t-\sqrt{-c}}{t+\sqrt{-c}}\right| + C_2$$

27–32. Initial value problems with complex roots *Find the general solution of the following differential equations. Then solve the given initial value problem.*

5.2.**32.** $y'' - y' + \frac{1}{2}y = 0;\ y(0) = 3,\ y'(0) = -2$

$1 = r^0$

$$r^2 - r + \frac{1}{2} = 0 \to r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1\cdot \frac{1}{2}}}{2\cdot 1}$$

$$= \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm \sqrt{-1}}{2}$$

$$= \frac{1}{2} \pm \frac{i}{2} \to$$

$$y(t) = c_1 e^{\left(\frac{1}{2}+\frac{i}{2}\right)t} + c_2 e^{\left(\frac{1}{2}-\frac{i}{2}\right)t}$$

$$= c_1 e^{\frac{1}{2}t}\sin\left(\tfrac{1}{2}t\right) + c_2 e^{\frac{1}{2}t}\cos\left(\tfrac{1}{2}t\right)$$

$$y'(t) = \left(c_1 e^{\frac{1}{2}t}\sin\left(\tfrac{1}{2}t\right) + c_2 e^{\frac{1}{2}t}\cos\left(\tfrac{1}{2}t\right)\right)'$$

$$= \left(c_1 e^{\frac{1}{2}t}\sin\left(\tfrac{1}{2}t\right)\right)' + \left(c_2 e^{\frac{1}{2}t}\cos\left(\tfrac{1}{2}t\right)\right)'$$

$$= \frac{1}{2}c_1 e^{\frac{1}{2}t}\sin(\tfrac{1}{2}t) + \frac{1}{2}c_1 e^{\frac{1}{2}t}\cos(\tfrac{1}{2}t) +$$

$$\frac{1}{2}c_2 e^{\frac{1}{2}t}\cos(\tfrac{1}{2}t) - \frac{1}{2}c_2 e^{\frac{1}{2}t}\sin(\tfrac{t}{2})$$

$$\rightarrow 3 = y(0) = c_1 \cdot 1 \cdot 0 + c_2 \cdot 1 \cdot 1 = c_2$$

$$-2 = y'(0) = \frac{1}{2}c_1 \cdot 1 \cdot 0 + \frac{1}{2}c_1 \cdot 1 \cdot 1$$

$$+ \frac{1}{2}c_2 \cdot 1 \cdot 1 - \frac{1}{2}c_2 \cdot 1 \cdot 0$$

$$= \frac{1}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}c_1 + \frac{3}{2}$$

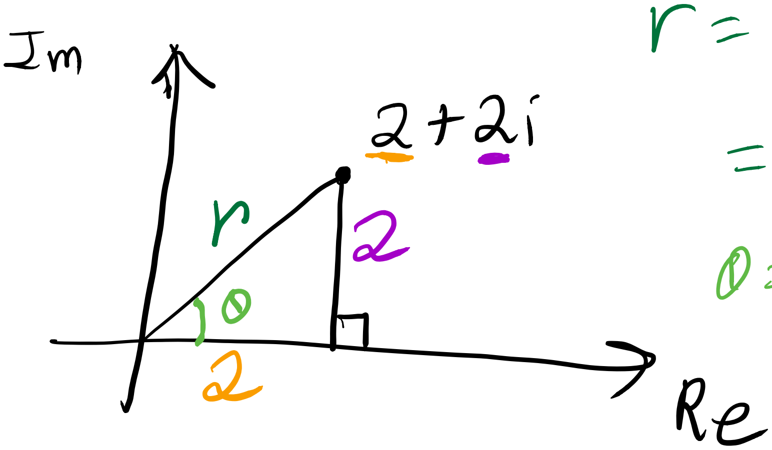
$$\rightarrow \frac{1}{2}c_1 = -2 - \frac{3}{2} = \frac{-7}{2}$$

$$\rightarrow c_1 = -7 \rightarrow$$

$$y(t) = -7e^{\frac{1}{2}t}\sin(\tfrac{1}{2}t) + 3e^{\frac{1}{2}t}\cos(\tfrac{1}{2}t)$$

47–54. Evaluating roots *Evaluate the following roots, being sure to include all possible values.* Then plot the values.
(From the appendix to chapter 5)

52. $(2+2i)^{1/3}$



$$r = \sqrt{2^2+2^2} = \sqrt{8}$$

$$= 2\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$$

$$\rightarrow 2+2i = 2\sqrt{2}\,e^{i\frac{\pi}{4}}$$

$$= 2\sqrt{2}\,e^{i\left(\frac{\pi}{4}+2\pi n\right)}$$

(n an integer)

$$(2+2i)^{1/3} = \left(2\sqrt{2}\,e^{i\left(\frac{\pi}{4}+2\pi n\right)}\right)^{\frac{1}{3}}$$

$$= \left(2\sqrt{2}\right)^{\frac{1}{3}} e^{i\frac{1}{3}\left(\frac{\pi}{4}+2\pi n\right)}$$

$$= \sqrt{2}\, e^{i\left(\frac{\pi}{12} + 2\pi \frac{n}{3}\right)}$$ n an integer

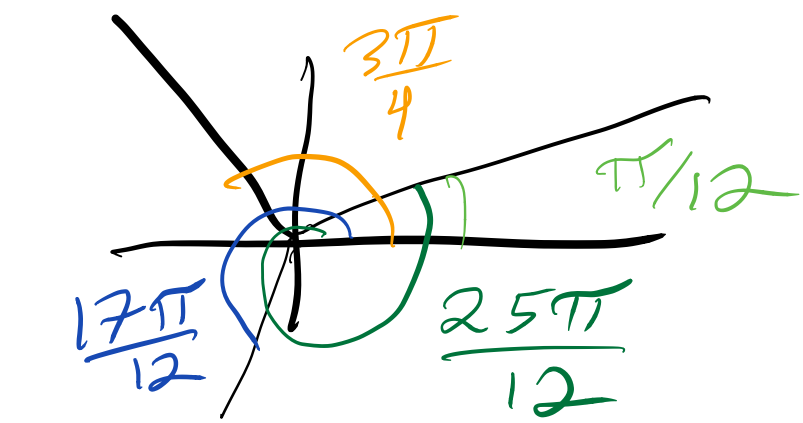
$$= \sqrt{2}\, e^{i\frac{\pi}{12}}$$ $n = 0$, 6, 9

$$\sqrt{2}\, e^{i\frac{3\pi}{4}}$$ $n = 1$, 4, 7, 10

$$\sqrt{2}\, e^{i\frac{17\pi}{12}}$$ $n = 2$, 5, 8, 11

$$\cancel{\sqrt{2}\, e^{i\frac{25\pi}{12}}}$$ $\cancel{n = 3}$

$$\cancel{\sqrt{2}\, e^{i\left(\frac{\pi}{12} + 2\pi\right)}}$$



$$\sqrt{2}\,\sqrt{2}\,\sqrt{2} = 2\sqrt{2}$$